Fermi's Golden Rule and H^1 Scattering for Nonlinear Klein-Gordon Equations with Metastable States

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In this talk, we will consider the following cubic Klein-Gordon equation with potentials (NLKG) in $\mathbb{R}^{3+1}$ for both focusing and defocusing case:

$$\partial_t^2 u - \Delta u + V(x)u + m^2 u = \lambda u^3, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0. \tag{1}$$

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \tag{2}$$

- We assume that $B^2 := -\Delta + V(x) + m^2$ has a positive eigenvalue Ω^2 and an eigenfunction $\psi(x)$

$$-\Delta\psi(x) + V(x)\psi(x) + m^2\psi(x) = \Omega^2\psi(x).$$

- Hence $e^{i\Omega t}\psi(x)$ solves linear Klein-Gordon equation (bound state).

Motivation

- Adding nonlinear perturbations, we will study the instability mechanism (**Fermi's Golden Rule**) for this bound state.
- We will show that the outcomes are anomalously slow-decaying waves (called **metastable states**).

Soffer-Weinstein's result

In 1999, Soffer and Weinstein first studied this problem and they proved the following result

Theorem I: Let $V(x)$ be real-valued and satisfy technical conditions:

- (V1) for $\delta > 5$ and $|\alpha| \leq 2$, $|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|^2)^{-\frac{\delta}{2}}$,
- (V2) $(-\Delta + 1)^{-1}\big((x \cdot \nabla)^l V(x)\big)(-\Delta + 1)^{-1}$ is bounded on L^2 for $|l| \leq N_*$ with $N_* \geq 10$.
- (V3) zero is not a resonance of the operator $-\Delta + V$.

Assume the operator

$$B^2 = -\Delta + V(x) + m^2 \tag{3}$$

has continuous spectrum, $\sigma_{cont}(B^2) = [m^2, +\infty)$, and a unique strictly positive simple eigenvalue, $\Omega^2 < m^2$ with associated normalized eigenfunction, ψ:

$$B^2 \psi = \Omega^2 \psi. \tag{4}$$

Soffer-Weinstein's result

Further assume: $0 < \Omega^2 < m^2 < 9\Omega^2$.
(This is related to a nonlinear damping mechanism (called *Fermi Golden Rule*)

$$\Gamma = \frac{\pi}{3\Omega}\left(P_c\psi^3, \delta(B - 3\Omega)P_c\psi^3\right) \equiv \frac{\pi}{3\Omega}|(\mathcal{F}_c\psi^3)(3\Omega)|^2 > 0. \tag{5}$$

Here, P_c denotes the projection onto the continuous spectral part of B and $\mathcal{F}_c$ denotes the Fourier transform relative to the continuous spectral part of B.)

Remark: Γ is the transition rate of energy from a bound state to free waves.

Soffer-Weinstein's result

Assume that the initial data u_0, u_1 are prescribed such that the norm $\|u_0\|_{W^{2,2}\cap W^{2,1}}$ and $\|u_1\|_{W^{2,2}\cap W^{2,1}}$ are sufficiently small. Then, the solution of the initial value problem for (1), with $\lambda \neq 0$ decays as $t \to +\infty$. Then solution $u(t,x)$ has the following expansion as $t \to +\infty$:

$$u(t,x) = 2\rho(t)\cos\theta(t)\psi(x) + \eta(t,x), \tag{6}$$

where

$$0 \leq \rho(t) \leq \frac{2^{\frac{1}{4}}\rho(0)}{\left(1 + \frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t\right)^{\frac{1}{4}}}, \tag{7}$$

$$\theta(t) - \Omega t = \mathcal{O}(t^{\frac{1}{2}}), \quad \text{and} \quad \|\eta(t,x)\|_{L^8_x(\mathbb{R}^3)} \leq \frac{1}{(1+t)^{\frac{3}{4}}}. \tag{8}$$

An-Soffer's result

Main theorems of An-Soffer are

Theorem II

Under the same assumption of Theorem I, we have

$$\frac{(1/2)^{\frac{1}{4}}\rho(0)}{\left(1+\frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t\right)^{\frac{1}{4}}} \leq \rho(t) \leq \frac{(3/2)^{\frac{1}{4}}\rho(0)}{\left(1+\frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t\right)^{\frac{1}{4}}}. \tag{9}$$

An-Soffer's result

Theorem III
Under the same assumption of Theorem I, there exist $S_1(x) \in L_x^2(\mathbb{R}^3)$ and $S_2(x) \in L_x^2(\mathbb{R}^3)$ such that as $t \to +\infty$

$$\|\eta(t,x) - \frac{\sin Bt}{B} S_1(x) - \cos Bt S_2(x)\|_{H_x^1(\mathbb{R}^3)} \to 0. \tag{10}$$

Remark: Since

$$\|2\rho(t)\cos\theta(t)\psi(x)\|_{H_x^1(\mathbb{R}^3)} \to 0, \tag{11}$$

as $t \to +\infty$. Therefore, we also prove
Theorem IV (H^1 Scattering) Under the same assumption of Theorem I, for solution $u(t,x)$ in (6), there exist $S_1(x) \in L_x^2(\mathbb{R}^3)$ and $S_2(x) \in L_x^2(\mathbb{R}^3)$ such that as $t \to +\infty$

$$\|u(t,x) - \frac{\sin Bt}{B} S_1(x) - \cos Bt S_2(x)\|_{H_x^1(\mathbb{R}^3)} \to 0. \tag{12}$$

Main difficulties

To prove Theorems II-IV, we would expect some difficulties. It is natural to ask

- Question 1: How to show the lower bound for $\rho(t)$ as in (9)?
- Question 2: We have $\|\eta(t,x)\|_{L^8_x} \leq 1/(1+t)^{\frac{3}{4}}$ from Soffer-Weinstein. But how to explore the detailed structures for $\eta(t,x)$ in L^2_x as in (10)?

Question 1

To derive a lower bound for $\rho(t)$, we use polar coordinates and introduce a new ODE approach. Under the ansatz (6)

$$u(t,x) = 2\rho(t)\cos\theta(t)\psi(x) + \eta(t,x),$$

and using a suitable gauge for NLKG, $\rho(t)$ would satisfy an ODE:

$$\rho'(t) + \frac{3\lambda^2}{4\Omega}\Gamma\rho^5 + \sum_{k\geq 1} c_k \|\psi\|_{L_x^4}^4 \rho(t)^5 \sin k\theta(t) + \sum_{k\geq 1} d_k \|\psi\|_{L_x^4}^4 \rho(t)^5 \cos k\theta(t)$$
$$+ \frac{\lambda}{\Omega}\|\psi\|_{L_x^4}^4 \rho^3 \sin 2\theta(t) + \frac{\lambda}{2\Omega}\|\psi\|_{L_x^4}^4 \rho^3 \sin 4\theta(t) + l.o.t = 0.$$

Here k are some positive integers; c_k and d_k are real numbers. And $\Gamma > 0$ is due to Fermi golden rule (5).

Question 1

This equation implies

$$\begin{aligned}\frac{1}{\rho(t)^4} =&\frac{(1+\frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t)}{\rho(0)^4} - \int_0^t 4c_k\|\psi\|_{L_x^4}^4 \sin k\theta(t')dt' \\ &- \int_0^t 4d_k\|\psi\|_{L_x^4}^4 \cos k\theta(t')dt' - \int_0^t \frac{4\lambda}{\Omega}\|\psi\|_{L_x^4}^4 \frac{\sin 2\theta(t')}{\rho(t')^2}dt' \\ &- \int_0^t \frac{2\lambda}{\Omega}\|\psi\|_{L_x^4}^4 \frac{\sin 4\theta(t')}{\rho(t')^2}dt' + l.o.t.\end{aligned} \tag{13}$$

In Theorem II, we hope to prove $1/\rho(t)^4 \approx 1+t$ for t large. This would give both lower and upper bounds for $\rho(t)$. Hence we only need to show that, on the right hand side of (13), the first term $(1+\frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t)/\rho(0)^4$ dominates.
Luckily, for $\theta(t)$ we have $\theta(t) = \Omega t + l.o.t.$ When t is large, the second and third terms $\int_0^t 4c_k\|\psi\|_{L_x^4}^4 \sin k\theta(t')dt'$ and $\int_0^t 4d_k\|\psi\|_{L_x^4}^4 \cos k\theta(t')dt'$ are like constants. They are much smaller than the first term.

Question 1

For $\int_0^t \frac{4\lambda}{\Omega}\|\psi\|_{L_x^4}^4 \sin 2\theta(t')/\rho(t')^2 dt'$ and $\int_0^t \frac{2\lambda}{\Omega}\|\psi\|_{L_x^4}^4 \sin 4\theta(t')/\rho(t')^2 dt'$, in principle we could use integration by part and hope to deal with terms as the second or the third term.

However, we encounter an additional difficulty: since we are proving upper and lower bounds at the same time, we cannot rule out the possibility that $\rho(t)$ decays faster than $1/(1+t)^{\frac{1}{4}}$. And this would make $1/\rho(t')^2$ out of control.

To overcome this difficulty, we construct a parametrix $\bar{\rho}(t) \geq 0$ through

$$\bar{\rho}(t)^4 = \frac{\rho(0)^4}{1 + \frac{3\lambda^2\Gamma}{\Omega}\rho(0)^4 t}. \tag{14}$$

Question 1

One can check $\bar{\rho}(t)$ satisfies:

$$\begin{aligned} \bar{\rho}'(t) &= -\frac{3\lambda^2}{4\Omega}\Gamma\bar{\rho}(t)^5, \\ \bar{\rho}(0) &= \rho(0). \end{aligned} \tag{15}$$

Here $\bar{\rho}(t)$ is an approximate solution for

$$\rho'(t) + \frac{3\lambda^2}{4\Omega}\Gamma\rho^5 + \sum_{k\geq 1} c_k\|\psi\|^4_{L^4_x}\rho(t)^5\sin k\theta(t) + \sum_{k\geq 1} d_k\|\psi\|^4_{L^4_x}\rho(t)^5\cos k\theta(t)$$
$$+\frac{\lambda}{\Omega}\|\psi\|^4_{L^4_x}\rho^3\sin 2\theta(t) + \frac{\lambda}{2\Omega}\|\psi\|^4_{L^4_x}\rho^3\sin 4\theta(t) + l.o.t = 0.$$

Then we introduce the unknown $\epsilon(t)$ through

$$\rho = \bar{\rho}(t)(1+\epsilon(t)). \tag{16}$$

For initial data, we have $\epsilon(0) = 0$. Therefore, the seeking of lower bound for $\rho(t)$ is reduced to close a bootstrap argument for $\epsilon(t)$ and to show that $\epsilon(t)$ is small for all the time. And this is proved in An-Soffer.

Remark: For this method, we do not use normal form transformation.

Question 2

After deriving the asymptotic behavior for $\rho(t)$, we move to study the L_x^2 norm of $\eta(t,x)$. For η, we have

$$\eta(t,x) = \eta_1(t,x) + \eta_2(t,x) + \eta_3(t,x),$$

where

$$(\partial_t^2 + B^2)\eta_1 = 0, \quad \eta_1(0,x) = P_c u_0, \quad \partial_t \eta_1(0,x) = P_c u_1,$$

$$(\partial_t^2 + B^2)\eta_2 = \lambda a^3 P_c \psi^3, \quad \eta_2(0,x) = 0, \quad \partial_t \eta_2(0,x) = 0, \tag{17}$$

$$(\partial_t^2 + B^2)\eta_3 = \lambda P_c(3a^2\psi^2\eta + 3a\psi\eta^2 + \eta^3), \quad \eta_3(0,x) = 0, \quad \partial_t \eta_3(0,x) = 0.$$

Here $a = a(t) := 2\rho(t)\cos\theta(t)$. Let's first focus on $\eta_2(t,x)$. From (17), we have

$$\eta_2(t,x) = \lambda \int_0^t \frac{\sin B(t-s)}{B} a^3(s) P_c \psi^3(x) ds. \tag{18}$$

Question 2

If we only use $|a(t)|^3 \le \rho(t)^3 \approx 1/(1+t)^{\frac{3}{4}}$ and the standard dispersive estimates for wave operator (see Theorem 2.1 in Soffer-Weinstein), we cannot even prove that $\eta_2(t,x) \in L^2_x$ for all $t \ge 0$.
We overcome this difficulty by constructing an auxiliary function $w(t,x)$ through solving

$$\begin{aligned}(i\partial_t + B)w =& \lambda a^3 P_c \psi^3, \\ w(0,x) =& 0.\end{aligned}$$

Hence

$$w(t,x) = -i\int_0^t e^{iB(t-s)}\lambda a^3(s) P_c \psi^3(x) ds.$$

Question 2

Thus,

$$\text{Im}\, w(t,x) = -\int_0^t \sin B(t-s)\, \lambda a^3(s) P_c \psi^3(x) ds,$$

and

$$\text{Re}\, w(t,x) = \int_0^t \cos B(t-s)\, \lambda a^3(s) P_c \psi^3(x) ds.$$

Define

$$l(t) := \|w(t,x)\|_{L_x^2}^2 = \|\text{Re}\, w(t,x)\|_{L_x^2}^2 + \|\text{Im}\, w(t,x)\|_{L_x^2}^2.$$

Question 2

Since B is a self-adjoint operator, a key cancellation happens and a simple calculation implies

$$\frac{d}{dt}l(t) = -2\int_{\mathbb{R}^3} \operatorname{Im} w(t,x)\lambda a^3(t)P_c\psi^3(x)dx.$$

Question 2

Using the definitions of $\mathrm{Re}\, u(t,x)$, $\mathrm{Im}\, u(t,x)$, together with the standard dispersive estimates for B and the fact $|a(t)| \leq 1/(1+t)^{\frac{1}{4}}$, we derive

$$\|\mathrm{Re}\, w(t,x),\ \mathrm{Im}\, w(t,x)\|_{L^8_x} \leq \frac{1}{(1+t)^{\frac{3}{4}}}.$$

With the estimate above, we arrive at

$$\begin{aligned} l(t) \leq& l(0) + \int_0^t |\frac{d}{dt} l(t)| dt \\ \leq& \int_0^t \|\mathrm{Im}\, w(t,x)\|_{L^8_x} |a^3(t)| \|P_c \psi^3(x)\|_{L_x^{\frac{8}{7}}} dt \\ \leq& \int_0^t \frac{1}{(1+t)^{\frac{3}{4}}} \cdot \frac{1}{(1+t)^{\frac{3}{4}}} < +\infty. \end{aligned}$$

Question 2

Therefore, we have proved

$$\text{Im}\, w(t,x) = \int_0^t \sin B(t-s)\, \lambda a^3(s) P_c \psi^3(x) ds \in L_x^2,$$

$$\text{Re}\, w(t,x) = \int_0^t \cos B(t-s)\, \lambda a^3(s) P_c \psi^3(x) ds \in L_x^2,$$

for any $t > 0$. Since $1/B$ is a bounded operator for L_x^2, we deduce

$$\eta_2(t,x) = \lambda \int_0^t \frac{\sin B(t-s)}{B} a^3(s) P_c \psi^3(x) ds \in L_x^2.$$

In order to prove Theorem III and Theorem IV, it requires more detailed analysis for $\eta_2(t,x)$ and $\eta_3(t,x)$ in L_x^2 norms. Details are shown in An-Soffer.

Some Historical Remarks

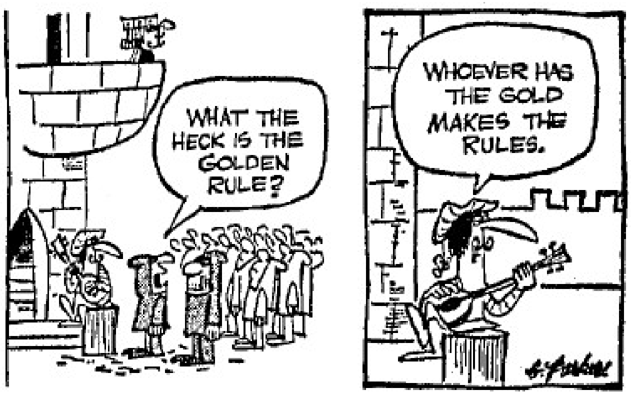
- In quantum mechanics, people observed some long-lived states, which last at least 100 to 1000 times longer than the expection. These long-lived states are called metastable states in physics literature. Mathematically, one would expect that these states carry anomalously slow-decaying rates.
- One way to produce a metastable state is through the instability of an excited state.

Some Historical Remarks

- To study the instability mechanism, with perturbation theory in 1927 Dirac did calculations in the following setting: Give two Hamiltonians H_0 and H_1 close to each each, assume they have eigenfunction (initial eigenstate) $i(x)$ and eigenfunction (final eigenstate) $f(x)$ respectively, Dirac calculated the transition probability per unit time from the state $i(x)$ to the state $f(x)$:

$$\Gamma_{i \to f} = \frac{4\pi^2}{h} \left| \int_{\mathbb{R}^3} i(x) H_1(x) f(x) dx \right|^2 \cdot \rho_f. \qquad (19)$$

  Here h is the Planck constant ($\approx 6.626 \times 10^{-34} \quad kg \cdot m^2 \cdot s^{-1}$) and ρ_f is the density of final states.
- In 1934 Fermi used (19) to establish his famous theory of beta decay. In nuclear physics, beta decay is a type of radioactive decay in which a $\beta-$ray (fast energetic electron or positron) and a neutrino are emitted from an atomic nucleus. In his paper, Fermi called (19) *golden rule*. Later, in physics community, (19) is called *Fermi's golden rule*.


WHAT THE HECK IS THE GOLDEN RULE?
WHOEVER HAS THE GOLD MAKES THE RULES.

Title: How to Make a Black Hole

Abstract: Black holes are predicted by Einstein's theory of general relativity, and now we have ample observational evidence for their existence. However theoretically there are many unanswered questions about how black holes come into being. In this talk, with tools from hyperbolic PDE, quasilinear elliptic equations, geometric analysis and dynamical systems, we will prove that, through a nonlinear focusing effect, initially low-amplitude and diffused gravitational waves can give birth to a black hole region in our universe. This result extends the 1965 Penrose's singularity theorem and it also proves a conjecture of Ashtekar on black hole thermodynamics.

THANK YOU!