Rigorous derivation of non-isentropic Low Mach number Navier-Stokes equations in bounded domains

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Part I. Introduction.

$1. Formal derivation of low Mach number limit.

Non-dimensional Full compressible Navier-Stokes equations

$$
\begin{aligned}
&\rho_t + \mathrm{div}(\rho u) = 0,\\
&(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \mathrm{div} u + \frac{1}{\epsilon^2}\nabla p = 0,\\
&c_v((\rho\Theta)_t + \mathrm{div}(\rho\Theta u)) + P\mathrm{div} u - \kappa\Delta\Theta = \epsilon^2(2\mu|D(u)|^2 + \lambda(\mathrm{div} u)^2).
\end{aligned}
$$

$\rho :$ density, $u = (u^1, ..., u^n) :$ velocity, $\Theta :$ temperature,

$p = R\rho\Theta :$ **pressure (for perfect gases)**,

$c_v, R > 0$: constant, $\kappa > 0 :$ heat conductivity constant,

$\mu > 0, \lambda \geq -\frac{2}{n}\mu :$ viscosity constants,

$\epsilon :$ **Mach number**<<1 (the fluid is "nearly incompressible")

Low Mach number expansion and limit:

compressible flows $\approx$ background incompressible flows

+ high order corrections(acoustic waves)

$$\begin{aligned} &\rho_\epsilon = \bar{\rho} + \epsilon\rho_1 + O(\epsilon^2); \\ &\Theta_\epsilon = \bar{\Theta} + \epsilon\Theta_1 + O(\epsilon^2); \\ &u_\epsilon = \bar{u} + O(\epsilon). \end{aligned}$$

Let $\epsilon \to 0 \Rightarrow \bar{\rho}\bar{\Theta}$=constant, and $(\bar{\rho}, \bar{u}, \pi)$ satisfies the following non-isentropic low Mach number Navier-Stokes equations:

$$\begin{aligned} &\bar{\rho}_t + \mathrm{div}(\bar{\rho}\bar{u}) = 0, \\ &(\bar{\rho}\bar{u})_t + \mathrm{div}(\rho\bar{u} \otimes \bar{u}) - \mu\Delta\bar{u} - \lambda\nabla\mathrm{div}\bar{u} + \nabla\pi = 0, \\ &\gamma\mathrm{div}\bar{u} = \mathrm{div}\left[\frac{\kappa}{R}\nabla\left(\frac{1}{\bar{\rho}}\right)\right], \quad \gamma = 1 + \frac{R}{c_v}. \end{aligned}$$

In particular, for **isentropic fluids** **or** $\bar{\rho} = const$ **or** $\kappa = 0$

The third equation $\Leftrightarrow \mathrm{div}\bar{u} = 0$ **(Incompressibility)**

Then the low Mach number limit reduces to the incompressible limit, where the limiting system is the homogeneous isentropic Navier-Stokes system (for the isentropic case, or the non-isentropic case with small temperature variation)

$$
\begin{aligned}
&\bar{u}_t + \bar{u} \cdot \nabla \bar{u} - \mu \Delta \bar{u} + \nabla \pi = 0, \\
&\mathrm{div}\bar{u} = 0.
\end{aligned}
$$

or inhomogeneous isentropic Navier-Stokes system (for the case of $\kappa = 0$ and large temperature variation)

$$
\begin{aligned}
&\bar{\rho}_t + \bar{u} \cdot \nabla \bar{\rho} = 0, \\
&(\bar{\rho}\bar{u})_t + \mathrm{div}(\rho \bar{u} \otimes \bar{u}) - \mu \Delta \bar{u} + \nabla \pi = 0, \\
&\mathrm{div}\bar{u} = 0,
\end{aligned}
$$

$2. Rigorous verification:

(1) Derive uniform estimates with respect to the Mach number $\epsilon \in (0, \bar{\epsilon}]$ in a time interval $[0, T]$ independent of ϵ

(2) Verify rigorously the limit as $\epsilon \to 0$.

\$3. Known results on low Mach number limit/ incompressible limit

1. **Incompressible limit of isentropic Navier-Stokes equations.**

   (a) The incompressible limit of *local-in-time smooth* solution with well-prepared initial data.

   ◇ S.Klainerman & A.Majda (1981): $\mathbb{R}^n$ or $\mathbb{T}^n$, $\mathrm{div} u_0 = 0$ or $\mathrm{div} u_0 = O(\epsilon)$ in high-norm (well-prepared initial data)
   Uniform estimates: using the anti-symmetric structure of singular differential operator
   Convergence: compactness theory

   (b) The incompressible limit of *global* weak solution for $t \in [0, T]$, with ill-prepared initial data ($\mathrm{div} u_0 = O(1)$).
   Uniform estimates: energy estimates of global weak solutions with finite energy which is developed by P.-L. Lions
   Convergence: using the “rescaled group method” which is developed by S. Schochet 1994 and E. Grenier 1997 independently

   ◇ P.-L. Lions & N. Masmoudi (1998): weak convergence, various boundary conditions, ill-prepared initial data.

$\diamond$ B. Desjardins & E. Grenier (1999): strong convergence in $\mathbb{R}^n$ (Strichartz estimates)

$\diamond$ Desjardins, Grenier, Lions, Masmoudi (1999); N. Jiang & N.Masmoudi (2015): strong convergence in bounded domains (spectrum method)
$\diamond$ Feireisl, et al (2011, 2013,...): strong convergence in exterior domains
$\diamond$ Feireisl, Novotny, Petzeltova, et al (2010,...): strongly stratified fluids, various boundary conditions

(c) The incompressible limit of *long-time* solution for $t \in [0, +\infty)$: difficult to get the uniform estimate for both $\epsilon \in (0, \bar{\epsilon}]$ and $t \in [0, +\infty)$.

$\diamond$ D. Hoff (1998):

weak solutions in $\mathbb{R}^3$ with compatible data, but require the global smoothness of the background incompressible solutions.

$\diamond$ H. Bessaih (1995), O.(2009), O. and D. Ren (2014)

strong solutions in bounded domains with & well-prepared initial data: difficult to get uniform high-norm estimates

(d) Unsolved problems: IBVP & ill-prepared initial data, free boundary problems, multi-scale limits, ...

2. **Non-isentropic Navier-Stokes equations**

The pressure variation $\approx \ \bar{p} + O(\epsilon)p_1(\rho, T)$ (for isentropic case, pressure variation $\approx$ density variation), which leads to the different structure on the singular differential operator and the difficulty in deriving the uniform estimates, in particular, for the case with solid boundary.

(a) Heat-conductivity coefficient $\kappa = 0$. (diffusive effect $<<$ convective effect)

- Bresch, Desjardins, Grenier & Lin (2002): show the formal expansion in periodic domains when the entropy is purely transported.
- Kim & Lee (2005): incompressible limit of strong solutions in R^3, with well-prepared data and large temperature variation.
- O. (2009, 2011); S. Jiang & O. (2011): incompressible limit in boundary domains, with well-prepared data and large temperature variation: uniform high-norm estimates

(b) **Heat-conductivity coefficient $\kappa > 0$: .**

- T. Alazard (2006): Low Mach number limit of *local-in-time* classical solution in $\mathbb{R}^n$, with ill-prepared data, for *perfect gas* $(P = R\rho T)$.
  Uniform estimates: the methods of pseudo-differential operators
  Convergence: due to Metivier and Schochet
- Feireisl, Novotny, Petzeltova, et al (2007, 2008,2010, 2013, ...): Low Mach number limit of global *variational* weak solution for Navier-Stokes-Fourier system with various boundary conditions and ill-prepared data and require that the temperature variation is small (but excludes the case of perfect gas, and the limiting velocity is divergence-free, that is, div$\bar{u}$=0)
  Convergence: rescaled linear group
- C. Dou, S. Jiang & O.(2015): Low Mach number limit of local strong solutions with well-prepared data for *perfect gas*, and require that the temperature variation is small: $T \approx 1 + O(\epsilon) \Rightarrow$ the limiting velocity: div$\bar{u}$=0. In this case, it is difficult to get the uniform estimates due to the solid boundary.

Question:

Low Mach number limit of local strong solution of full Navier-Stokes equations for perfect gases ($\kappa > 0$) in bounded domains in case of large temperature variation? ($\gamma \mathrm{div} \bar{u} = \mathrm{div}\left[\frac{\kappa}{R}\nabla\left(\frac{1}{\bar{\rho}}\right)\right]$)

Difficulties of this problem

- The system is nonlinear and strongly coupled.
- The singular differential operator of $O(\frac{1}{\epsilon})$ is no longer anti-symmetric (after the transform).
- Unlike the cases of the whole space or the torus, integrating by parts cannot be applied for the estimates of high-order spatial derivatives.

Part II. Incompressible limit of local strong solutions for full Navier-Stokes equations in 3-D bounded domains (Joint work with Qiangchang Ju (IAPCM, Beijing)).

In what follows, we drop the superscript ϵ for simplicity. We introduce the change of variables:

$$p = 1 + \epsilon q, \quad \Theta = 1 + \theta \ (\textit{large temperature variation}),$$

where the pressure p satisfies

$$p_t + \mathbf{u} \cdot \nabla p + 2p\mathrm{div}\mathbf{u} = \kappa \Delta \Theta + \epsilon^2 [2\mu |D(\mathbf{u})|^2 + \lambda (\mathrm{div}\mathbf{u})^2], \tag{1}$$

by using the density equation and the temperature equation, and the pressure law $p = R\rho\Theta$.

Then the full Navier-Stokes equations are converted into into an “overdetermined” system

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \dfrac{1}{\epsilon} \nabla q = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \mathrm{div} \mathbf{u}, \\ \rho(\theta_t + \mathbf{u} \cdot \nabla \theta) + (1 + \epsilon q) \mathrm{div} \mathbf{u} = \kappa \Delta \theta + \epsilon^2 [2\mu |D(\mathbf{u})|^2 + \lambda (\mathrm{div} \mathbf{u})^2], \\ q_t + \mathbf{u} \cdot \nabla q + 2q \mathrm{div} \mathbf{u} + \dfrac{2}{\epsilon} \mathrm{div} \mathbf{u} - \dfrac{\kappa}{\epsilon} \Delta \theta = \epsilon [2\mu |D(\mathbf{u})|^2 + \lambda (\mathrm{div} \mathbf{u})^2], \end{cases} \tag{2}$$

where any of these equations coincides with the others. Suppose that $(\rho, \mathbf{u}, q, \theta)$ satisfies the following initial and boundary conditions:

$$(\rho, \mathbf{u}, q, \theta)|_{t=0} = (\rho_0, \mathbf{u}_0, q_0, \theta_0) := (\rho_0, \mathbf{u}_0, \frac{1}{\epsilon}(\rho_0 \Theta_0 - 1), \Theta_0 - 1). \tag{3}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{n} \times \mathrm{curl} \mathbf{u}|_{\partial\Omega} = 0, \ \frac{\partial \theta}{\partial n}\Big|_{\partial\Omega} = 0. \tag{4}$$

where $\Omega \subset \mathbb{R}^3$ is a simply connected, bounded domain with C^4-boundary $\partial\Omega$.

We introduce the change of variable:

$$\mathbf{v} = \mathbf{u} - \frac{k}{2}\nabla\theta. \tag{5}$$

Then $(\rho, \mathbf{v}, q, \theta)$ satisfies the following initial-boundary value problem

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, & \text{in} \quad \Omega \times (0,T], \\ \rho(\mathbf{v}_t + \mathbf{u}\cdot\nabla\mathbf{v}) + \dfrac{1}{\epsilon}\nabla q - \mu\Delta\mathbf{v} - \xi\nabla\mathrm{div}\mathbf{v} - \beta\nabla\Delta\theta = f, & \text{in} \quad \Omega \times (0,T], \\ \dfrac{1}{2}(q_t + \mathbf{u}\cdot\nabla q) + \dfrac{1}{\epsilon}\mathrm{div}\mathbf{v} = g, & \text{in} \quad \Omega \times (0,T], \\ \rho(\theta_t + \mathbf{u}\cdot\nabla\theta) - \dfrac{3\kappa}{2}\Delta\theta + \mathrm{div}\mathbf{v} = h, & \text{in} \quad \Omega \times (0,T], \\ \mathbf{v}\cdot\mathbf{n} = 0, \ \ \mathbf{n}\times\mathrm{curl}\mathbf{v} = 0, \ \dfrac{\partial\theta}{\partial n} = 0, & \text{on} \quad \partial\Omega \times (0,T], \\ (\rho, \mathbf{v}, q, \theta)|_{t=0} = (\rho_0, \mathbf{v}_0, q_0, \theta_0)(x), & x \in \Omega, \end{cases}$$

(6)

where the constants $\xi := \mu + \lambda + \frac{\kappa}{2}$, $\beta := \frac{\kappa}{2}(2\mu + \lambda - \frac{3}{2}\kappa)$, and the functions $\mathbf{v}_0 := \mathbf{u}_0 - \frac{\kappa}{2}\nabla\theta_0$,

$$
\begin{aligned}
f &:= \frac{\kappa}{2}[\nabla\rho(\theta_t + \mathbf{u}\cdot\nabla\theta) + \rho\nabla\mathbf{u}\nabla\theta - \nabla h], \\
g &:= -q\mathrm{div}\mathbf{u} + \frac{\epsilon}{2}[2\mu|D(\mathbf{u})|^2 + \lambda(\mathrm{div}\mathbf{u})^2], \\
h &:= -\epsilon q\mathrm{div}\mathbf{u} + \epsilon^2[2\mu|D(\mathbf{u})|^2 + \lambda(\mathrm{div}\mathbf{u})^2].
\end{aligned} \tag{7}
$$

Although the singular differential operators of $(6)_{2,3}$ of the order $\frac{1}{\epsilon}$ are in anti-symmetric form in the current formulation, however, the third order differential operator is introduced in $(6)_2$, which creates essential difficulty in the uniform estimates.

The aims of our problem:

(1) Derive uniform estimates with respect to the Mach number $\epsilon \in (0, \bar{\epsilon}]$ and the time $t \in [0, T])$ for some $\bar{\epsilon}$ and T.

(2) Verify rigorously the singular limit as Mach number $\epsilon \to 0$.

Theorem 1. (Local existence) *Let $\epsilon \in (0,1]$ be fixed and $\Omega \subset \mathbb{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Suppose that the initial datum $(\rho_0^\epsilon, \mathbf{v}_0^\epsilon, q_0^\epsilon) \in H^3(\Omega)$, $\theta_0^\epsilon \in H^4(\Omega)$ satisfies $\rho_0^\epsilon \geq m > 0$ and $1 + \theta_0^\epsilon \geq m$ for some constant m, and*

$$(\partial_t^k \rho^\epsilon(0), \partial_t^k \mathbf{v}^\epsilon(0), \partial_t^k q^\epsilon(0)) \in H^{2-k}(\Omega),\ \partial_t^k \theta^\epsilon(0) \in H^{3-k}(\Omega), \quad k = 0, 1.$$

Assume the following compatibility conditions are satisfied:

$$\mathbf{v}_0^\epsilon \cdot n = \mathbf{v}_t^\epsilon(0) \cdot n = 0,\ n \times \mathrm{curl}\mathbf{v}_0^\epsilon = n \times \mathrm{curl}\mathbf{v}_t^\epsilon(0) = 0 \quad \text{on} \ \ \partial\Omega, \tag{8}$$

$$\frac{\partial \theta_0^\epsilon}{\partial n} = \frac{\partial \theta_t^\epsilon(0)}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{9}$$

Then there exists a positive constant $T^\epsilon = T^\epsilon(\rho_0^\epsilon, \mathbf{v}_0^\epsilon, q_0^\epsilon, \theta_0^\epsilon, m, \epsilon)$, such that the initial-boundary problem (6) *admits a unique solution $(\rho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon, \theta^\epsilon)$,* *satisfying* $\rho^\epsilon > 0$ *and* $1+\theta^\epsilon > 0$ *in* $\Omega \times (0, T^\epsilon)$ *and*

$$
\begin{aligned}
&(\partial_t^k \rho^\epsilon, \partial_t^k \mathbf{v}^\epsilon, \partial_t^k q^\epsilon) \in C([0,T^\epsilon], H^{2-k}),\ \partial_t^k \mathbf{v}^\epsilon \in L^2(0,T^\epsilon; H^{3-k}),\\
&\partial_t^k \theta^\epsilon \in C([0,T^\epsilon], H^{3-k}) \cap L^2(0,T^\epsilon; H^{4-k}), \qquad k=0,1.
\end{aligned}
$$

Definition 1. *We define the uniform energy*

$$
\begin{aligned}
M^\epsilon(t) :=& \|(\rho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon)\|^2_{L^\infty(0,t;H^2(\Omega))} + \|\theta^\epsilon\|^2_{L^\infty(0,t;H^3(\Omega))} + \|(\rho^\epsilon)^{-1}\|^2_{L^\infty_{x,t}}\\
&+ \|(\rho^\epsilon_t, \mathbf{v}^\epsilon_t, q^\epsilon_t)\|^2_{L^\infty(0,t;L^2(\Omega))} + \|\theta^\epsilon_t\|^2_{L^\infty(0,t;H^1(\Omega))}\\
&+ \|(\mathbf{v}^\epsilon, \nabla\theta^\epsilon)\|^2_{L^2(0,t;H^3(\Omega))} + \|(\mathbf{v}^\epsilon_t, \nabla\theta^\epsilon_t)\|^2_{L^2(0,t;H^1(\Omega))}\\
&+ \|\epsilon(\rho^\epsilon_t, \mathbf{v}^\epsilon_t, q^\epsilon_t, \nabla\theta^\epsilon_t)\|^2_{L^\infty(0,t;H^1(\Omega))} + \|\epsilon(\mathbf{v}^\epsilon_t, \nabla\theta^\epsilon_t)\|^2_{L^2(0,t;H^2(\Omega))}.
\end{aligned}
$$

Remark 1. *We can derive from* (5) *and* $(2)_2$, *and the definition of* $M^\epsilon(t)$ *that, for any* $t \geq 0$,

$$\begin{aligned}
&\|\mathbf{u}^\epsilon\|^2_{L^\infty(0,t;H^2(\Omega))} + \|\mathbf{u}^\epsilon_t\|^2_{L^\infty(0,t;L^2(\Omega))} + \|\mathbf{u}^\epsilon\|^2_{L^2(0,t;H^3(\Omega))} \\
&\qquad + \|\mathbf{u}^\epsilon_t\|^2_{L^2(0,t;H^1(\Omega))} \leq CM^\epsilon(t),
\end{aligned}$$

$$\begin{aligned}
&\|\epsilon\mathbf{u}^\epsilon\|^2_{L^\infty(0,t;H^3(\Omega))} + \|\epsilon\mathbf{u}^\epsilon_t\|^2_{L^\infty(0,t;H^1(\Omega))} \\
&\qquad + \|\epsilon\mathbf{u}^\epsilon_t\|^2_{L^2(0,t;H^2(\Omega))} \leq C[(M^\epsilon(t))^2 + (M^\epsilon(t))^3].
\end{aligned}$$

We first show the following energy estimates which will give the uniform estimates.

Proposition 1. *For any* $t \in (0,1]$ *and* $\epsilon \in (0,1]$, *we have*

$$M^\epsilon(t) \leq C_0(M^\epsilon_0)\exp\{C_1(t^{\frac{1}{4}} + \epsilon)(M^\epsilon(t))^4\}, \tag{10}$$

where $C_0(\cdot)$ *is a positive and continuous function and* C_1 *is a positive constant, and both of them are independent of* $M^\epsilon(t)$ *and* $M^\epsilon_0 := M^\epsilon(0)$.

Theorem 2. (Uniform estimates) *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C^4$ and $\beta > 0$. Assume that $M_0^\epsilon \leq \alpha$, and $1 + \theta_0^\epsilon \geq A$, where α and A are positive constants independent of ϵ. Then $\exists$ positive constants $\epsilon_0 := \epsilon_0(\Omega, \alpha, A)$, $T := T(\Omega, \alpha, A)$ and $C := C(\epsilon_0, T)$ independent of ϵ, such that* (6) *$\exists!$ $(\rho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon, \theta^\epsilon)$ in $\Omega \times (0, T]$, with*

$$M^\epsilon(t) \leq C, \quad \epsilon \in (0, \epsilon_0],\ t \in (0, T], \tag{11}$$

$$1 + \theta^\epsilon(x, t) \geq C^{-1}, \quad (x, t) \in \Omega \times [0, T]. \tag{12}$$

Remark 2. *To ensure that $M_0^\epsilon \leq \alpha$, it suffices to suppose that $\|(\rho_0^\epsilon, \mathbf{v}_0^\epsilon, q_0^\epsilon)\|_{H^2(\Omega)} + \|\theta_0^\epsilon\|_{H^3(\Omega)} \leq C$, $\epsilon(\|(\rho_0^\epsilon, \mathbf{v}_0^\epsilon, q_0^\epsilon)\|_{H^3(\Omega)} + \|\theta_0^\epsilon\|_{H^4(\Omega)}) \leq C$, $\rho_0^\epsilon \geq C^{-1}$ and*

$$\|\mathrm{div}\mathbf{v}_0^\epsilon\|_{L^2} + \|\nabla q_0^\epsilon\|_{L^2} \leq C\epsilon,$$

for some constant $C > 0$ independent of ϵ. The last inequality is the minimal requirement on the incompressibility of the initial data.

Remark 3. *$\beta > 0$: thermal diffusion $<$ viscosity effect.*

Using the compactness theory, we canverify the following results.

Theorem 3. (Low Mach number limit) *Let $(\rho^\epsilon, \mathbf{v}^\epsilon, q^\epsilon, \theta^\epsilon)$ be the solution of* (6)*, satisfying the uniform-in-ϵ estimates established in Theorem 1, and $\mathbf{u}^\epsilon = \mathbf{v}^\epsilon - \frac{\kappa}{2}\nabla\theta^\epsilon$. Then $(\rho^\epsilon, \mathbf{u}^\epsilon, \theta^\epsilon) \rightharpoonup (\theta^{-1}, \omega, \theta)$ weakly-$*$ in $[L^\infty(0,T;H^2(\Omega))]^2 \times L^\infty(0,T;H^3(\Omega))$ as $\epsilon \to 0$, moreover, there exists a function $\pi(x,t)$, such that (ω, θ, π) satisfies the following initial-boundary value problem of low Mach number model:*

$$\begin{cases} \mathrm{div}\omega = \dfrac{\kappa}{2}\triangle\theta, & \text{in} \quad \Omega \times (0,T], \\ \theta^{-1}(\omega_t + \omega\cdot\nabla\omega) + \nabla\pi = \mu\Delta\omega + (\mu+\lambda)\nabla\mathrm{div}\omega, & \text{in} \quad \Omega \times (0,T], \\ \theta^{-1}(\theta_t + \omega\cdot\nabla\theta) = \dfrac{3\kappa}{2}\Delta\theta, & \text{in} \quad \Omega \times (0,T], \\ \omega\cdot n = 0,\ n\times \mathrm{curl}\omega = 0,\ \dfrac{\partial\theta}{\partial n} = 0, & \text{on} \quad \partial\Omega \times [0,T], \\ (\omega,\theta)|_{t=0} = (\omega_0,\theta_0)(x), & x \in \Omega, \end{cases}$$

where (ω_0, θ_0) is the weak limit of $(\mathbf{u}_0^\epsilon, \theta_0^\epsilon)$ in $H^2(\Omega)\times H^3(\Omega)$.

Difficulties (compared with previous results):

- The singular differential operator of $O(\frac{1}{\epsilon})$ is no longer anti-symmetric after the transform $p = 1 + \epsilon q, \quad \Theta = 1 + \theta$, and the third order term $-\beta\nabla\Delta\theta$ appears in the momentum equation after the change of variable $\mathbf{v} = \mathbf{u} - \frac{k}{2}\nabla\theta$. Thus the classical strategy cannot apply.

- The unsigned integrals of highest order spatial and time derivatives need to be absorbed carefully in the energy estimates.

Crucial points:

- Determine the uniform energy $M^\epsilon(t)$

- Carry out the estimates of high-order (weighted) estimates

- Cancel the unsigned integrals

Strategy of uniform estimates:

1. Energy estimates of the density ρ^ϵ and weighted estimates of ρ_t^ϵ

2. Low estimates and weighted high order estimates of q_t^ϵ and v_t^ϵ

3. Energy estimates of $\mathrm{curl} v^\epsilon = \mathrm{curl} u^\epsilon$ and weighted estimates of $\mathrm{curl} u_t^\epsilon$

4. High order estimates and weighted high order estimates of θ_t^ϵ
   (the order of θ_t^ϵ= the order of v_t^ϵ $+1$)

5. High order spatial estimates of q^ϵ, v^ϵ and θ^ϵ.

Part III. Sketch of the proof

1. Energy estimates of the density ρ^ϵ and weighted estimates of ρ_t^ϵ.

Lemma 1. *For any $t \in [0,T]$, we have*

$$\|\rho^{-1}\|^2_{L^\infty_{x,t}} + \|\rho\|^2_{L^\infty_t(H^2)} + \|\rho_t\|^2_{L^\infty_t(L^2)} + \|\epsilon\rho_t\|^2_{L^\infty_t(H^1)} \lesssim M_0 \exp\left\{Ct^{\frac{1}{2}}M(t)\right\}. \tag{13}$$

It is done by the method of characteristics and the energy method (no singular term occurs in the density equation).

2. Low estimates and weighted high order estimates of q_t^ϵ and v_t^ϵ.

Lemma 2. *For any $t \in [0,T]$, we have*

$$\begin{aligned}&\|(\sqrt{\rho}q_t, \mathbf{v}_t)\|^2_{L_t^\infty(L^2)} + \|\mathbf{v}_t\|^2_{L_t^2(H^1)} \\ &\qquad \lesssim C_0(M_0) + t^{\frac{1}{4}}M^2(t) - \beta\int_0^t\int_\Omega \Delta\theta_t \mathrm{div}\mathbf{v}_t dxdt.\end{aligned} \tag{14}$$

Lemma 3. *For any $t \in [0,T]$ and $\epsilon \in (0,1]$, we have*

$$\begin{aligned}&\|\epsilon(\sqrt{\rho^{-1}}\nabla q_t, \mathrm{div}\mathbf{v}_t)\|^2_{L_t^\infty(L^2)} + \|\epsilon\mathrm{div}\mathbf{v}_t\|^2_{L_t^2(H^1)} \\ &\qquad \lesssim C_0(M_0)\exp\left\{C(t^{\frac{1}{4}}+\epsilon)M^4(t)\right\} - \beta\epsilon^2\int_0^t\int_\Omega \rho^{-1}\nabla\Delta\theta_t \cdot \nabla\mathrm{div}\mathbf{v}_t dxdt.\end{aligned} \tag{15}$$

The terms in blue are unsigned highest order terms which cannot be absorbed by the left-hand side.

3. Energy estimates of $\mathrm{curl} v^\epsilon$ and weighted estimates of $\mathrm{curl} v_t^\epsilon$.

Note that $\mathbf{v} = \mathbf{u} - \frac{\kappa}{2}\nabla\theta$ and $\mathrm{curl}\nabla = 0$. To estimate $\mathrm{curl}\mathbf{v}$ and its spatial and time derivatives, it suffices to evaluate $\mathrm{curl}\mathbf{u}$ and its derivatives.

Applying the operator "curl" on $(2)_2$ and set $\mathbf{w} := \mathrm{curl}\mathbf{v} = \mathrm{curl}\mathbf{u}$, we obtain

$$\rho(\mathbf{w}_t + \mathbf{u}\cdot\nabla\mathbf{w}) - \mu\Delta\mathbf{w} = \mathcal{F}, \quad \text{in} \quad \Omega\times(0,T], \tag{16}$$

where $\mathcal{F} := \mathcal{F}_1 + \mathcal{F}_2 := [\rho, \mathrm{curl}]\mathbf{u}_t + [\rho\mathbf{u}\cdot\nabla, \mathrm{curl}]\mathbf{u}$ with [a,b]:=ab-ba. Note that the above equation is associated with the boundary condition

$$\mathbf{n}\times\mathbf{w}|_{\partial\Omega\times[0,T]} = 0. \tag{17}$$

Lemma 4. *For any $t\in[0,T]$, we have*

$$\begin{aligned}\|\mathrm{curl}\mathbf{u}\|^2_{L^\infty_t(H^1)} + \|\epsilon\,\mathrm{curl}\mathbf{u}_t\|^2_{L^\infty_t(L^2)} + \|\mathrm{curl}\mathbf{u}\|^2_{L^2_t(H^2)} + \|\epsilon\,\mathrm{curl}\mathbf{u}_t\|^2_{L^2_t(H^1)}\\ \lesssim C_0(M_0)\exp\left\{Ct^{\frac14}M^3(t)\right\}.\end{aligned} \tag{18}$$

4. High order estimates and weighted high order estimates of θ_t^ϵ.
(the order of θ_t^ϵ= the order of v_t^ϵ +1)
Lemma 5. *For any* $t \in [0,T]$ *and* $\epsilon \in (0,1]$, *we have*

$$\begin{aligned}\|(\sqrt{\rho}\theta_t,\nabla\theta_t)\|^2_{L_t^\infty(L^2)} + \|(\nabla\theta_t,\Delta\theta_t)\|^2_{L_t^2(L^2)} \\ \lesssim M_0 + (t^{\frac{1}{4}}+\epsilon)M^3(t) + \int_0^t\int_\Omega \mathrm{div}\mathbf{v}_t\Delta\theta_t dxdt.\end{aligned} \tag{19}$$

Lemma 6. *For any* $t \in [0,T]$ *and* $\epsilon \in (0,1]$, *we have*

$$\begin{aligned}\|\epsilon\Delta\theta_t\|^2_{L_t^\infty(L^2)} + \|\epsilon\nabla\Delta\theta_t\|^2_{L_t^2(L^2)} \\ \lesssim C_0(M_0)\exp\left\{C(t^{\frac{1}{4}}+\epsilon)M^{\frac{7}{2}}(t)\right\} + \epsilon^2\int_0^t\int_\Omega \rho^{-1}\nabla\Delta\theta_t\cdot\nabla\mathrm{div}\mathbf{v}_t dxdt.\end{aligned} \tag{20}$$

Now we can balance the extra terms of high order derivatives in Step 2.

5. High order spatial estimates of q^ϵ, v^ϵ and θ^ϵ.

Lemma 7. *For any $t \in [0, T]$ and $\epsilon \in (0, 1]$, we have*

$$\|\mathbf{v}\|^2_{L^\infty_t(H^2)} \lesssim C_0(M_0) \exp\left\{C(t^{\frac{1}{4}} + \epsilon) M^4(t)\right\}, \tag{21}$$

$$\begin{aligned}&\|\nabla^2 q\|^2_{L^\infty_t(L^2)} + \|\nabla^2 \mathrm{div}\mathbf{v}\|^2_{L^2_t(L^2)} \\ &\qquad \lesssim M_0 + (t^{\frac{1}{8}} + \epsilon) M^3(t) - \beta \int_0^t \int_\Omega \partial_i \nabla \Delta\theta \cdot \partial_i \nabla \mathrm{div}\mathbf{v} dxdt.\end{aligned} \tag{22}$$

Lemma 8. *For any $t \in [0, T]$, we have*

$$\|\Delta \nabla \theta\|^2_{L^\infty_t(L^2)} \lesssim M_0 + t^{\frac{1}{2}} M^{\frac{3}{2}}(t), \tag{23}$$

$$\begin{aligned}\|\partial_i \nabla \Delta\theta\|^2_{L^2_t(L^2)} \leq & C_0(M_0) \exp\{(t^{\frac{1}{4}} + \epsilon) M^3(t)\} \\ & + \int_0^t \int_\Omega \partial_i \nabla \Delta\theta \cdot \partial_i \nabla \mathrm{div}\mathbf{v} dxdt.\end{aligned} \tag{24}$$

Part IV. Future Works.

- The (multi-scale) singular limit of 3-D full N-S equations in bounded domains.
- Low Mach number limit of isentropic/non-isentropic N-S equations in 3-D bounded domains with ill-prepared initial data (in preparation).
- The singular limits of multi-dimensional free boundary problems of N-S equations (in preapration).

Thank you!