Aggregation equation with diffusion: uniqueness/non-uniqueness of steady states

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Aggregation equation with (degenerate) diffusion

- In this talk, we consider

$$\rho_t = \underbrace{\Delta \rho^m}_{\text{local repulsion}} + \underbrace{\nabla \cdot (\rho \nabla (W * \rho))}_{\text{nonlocal interaction}} \quad \text{in } \mathbb{R}^d,$$

  where $m \geq 1$, W is radially symmetric, and $W(r)$ is increasing. (So W is an attractive interaction potential).

- An example is the Keller-Segel equation, which models the collective motion of cells attracted by a self-emitted chemical substance:

$$\rho_t = \Delta \rho + \nabla \cdot (\rho \nabla (\mathcal{N} * \rho)),$$

  where $\mathcal{N} = \frac{1}{2\pi} \log |x|$ is the Newtonian potential in $\mathbb{R}^2$.

- The nonlinear diffusion term with $m > 1$ models the anti-overcrowding effect.

  (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06)

Free energy functional

- The associated free energy functional plays an important role:

$$E[\rho] = \underbrace{\frac{1}{m-1}\int \rho^m dx}_{=:S[\rho] \text{ (entropy)}} + \underbrace{\frac{1}{2}\int \rho(\rho * W) dx}_{=:I[\rho] \text{ (interaction energy)}} .$$

(When $m = 1$, the first term becomes $\int \rho \log \rho dx$).

- Formally taking time derivatives along a solution, we have

$$\frac{d}{dt}E[\rho] = -\int \rho \left|\nabla(\frac{m}{m-1}\rho^{m-1} + \rho * W)\right|^2 dx \le 0.$$

- Formally, the solution is a gradient flow of E in the metric space endowed by the 2-Wasserstein distance. (But rigorously justifying this requires certain convexity of W).
(Ambrosio-Gigli-Savare '08)

Main questions

In order to understand the long-time dynamics, a key step is to identify the stationary solutions.

Questions

1. For a given mass, does there exist a stationary solution?
2. Are they necessarily radially symmetric (up to a translation)?
3. If so, is it unique within the radial class?
4. If so, are they global attractors for the evolution equation?

- Existence of stationary solution can be done by a concentration-compactness argument (Lions '84):
  - For power-law kernels $W = |x|^k/k$, there exists a global minimizer when $m > 1 - k/d$.
  - For $m > 2$, there exists a global minimizer for any attractive kernel (Bedrossian '11)
  - For $1 \leq m < 2$, criteria of existence v.s. non-existence are given in Carrillo–Delgadino–Patacchini '18.

Symmetric or not?

By Riesz rearrangement inequality, a global minimizer of E must be radially decreasing. But can there be other stationary solutions?

Questions

Must stationary solutions be radially symmetric (up to a translation)?

Here a stationary solution satisfies

$$\frac{m}{m-1}\rho^{m-1} + W * \rho = C_i \quad \text{in } \operatorname{supp} \rho.$$

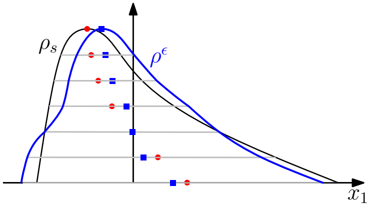
For all attractive kernels W that is no more singular than Newtonian kernel, we give a positive answer:

Theorem (Carrillo-Hittmeir-Volzone-Y. '16, to appear in *Invent. Math.*)

Let $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a stationary solution in the above sense. Then ρ_s must be radially decreasing up to a translation.

Sketch of the symmetry proof

- Idea: Assume a stationary solution ρ_s is non-radial, we perturb it using its continuous Steiner symmetrization:



- Since $\int \rho_s^m = \int (\rho^\epsilon)^m$, and interaction energy decreases in the first order for a short time (need some work to check this!),

$$E_m[\rho^\epsilon] - E_m[\rho_s] < -c\epsilon \quad \text{for all sufficiently small } \epsilon > 0,$$

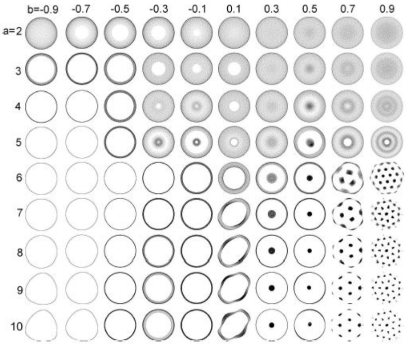
where $c > 0$ depending on ρ_s and W.

- If the equation has a rigorous gradient flow formulation, the above argument implies that $|\partial E|[\rho_s] \geq c$, directly leading to a contradiction.

Contrast with the attractive-repulsive kernel

If W is repulsive in short-range and attracting in long-range, then stationary solutions to $\rho_t = \nabla \cdot \big(\rho \nabla (W * \rho)\big)$ can have many non-radial patterns.

For example, when $W'(r) = \tanh((1-r)a) + b$ with parameters a, b, below are the patterns of stationary solutions for some a, b: (Kolokolnikov-Sun-Uminsky-Bertozzi, '11)



Unique or not?

Now that all stationary solutions are known to be radially decreasing (up to a translation), a natural question is whether there is uniqueness within this class.

Questions

For attractive kernels, for a given mass, must stationary solutions be unique?

Uniqueness results are only known in the following cases:

- $W = \mathcal{N}$ is the Newtonian potential in $\mathbb{R}^d$, and m is in the diffusion dominated regime. (Lieb–Yau '87)
- $W = \mathcal{N} * h$, where $h \geq 0$ is radially decreasing. (Kim–Yao '12)
- W is an attractive Riesz potential, and m is in the diffusion dominated regime. (Carrillo–Hoffmann–Mainini–Volzone '18, Calvez–Carrillo–Hoffmann '19)
- $m = 2$ and W is a C^2 attractive potential. (Burger–Di Francesco–Franek '13 and Kaib '17)

Theorem (Delgadino–Yan–Y., '19)

Let $m \geq 2$ and $W \in C^1(\mathbb{R}^d \setminus \{0\})$ be an attractive potential with $W'(r) \lesssim r^{-d-1+\delta}$ for some $\delta > 0$ for all $r \in (0,1)$. Then for any given mass, there is at most one stationary solution in $L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ up to a translation.

Idea of proof (when the gradient flow structure is rigorous):

- Due to the gradient flow structure, a stationary solution ρ_s should be a critical point of the energy functional E.
- For $m \geq 2$, if ρ_0, ρ_1 are two radial stationary solutions with the same mass, we will construct a curve $\{\rho_t\}_{t=0}^1$ connecting them, such that the energy along this curve is strictly convex.
- Therefore ρ_0 and ρ_1 can't be both critical points!

But how to find such an interpolation curve?

Convexity along interpolation curve

So far convexity along an interpolation curve are known only for the following kernels:

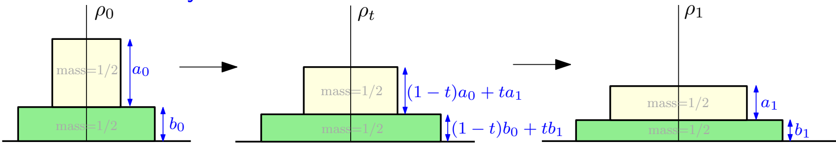
- If W is convex, energy is convex along the geodesic in 2-Wasserstein metric.
- If $\hat{W}(\xi) \geq 0$ for all ξ, energy is convex along the linear interpolation in ρ_0 and ρ_1.
- But a general attractive kernel does not satisfy either property.

Given any two radially decreasing ρ_0, ρ_1, we construct a novel interpolation curve $\{\rho_t\}_{t=0}^1$, such that:

1. $I[\rho] = \int \rho(\rho * W)dx$ is strictly convex along the curve for all attractive potential W.
2. $S[\rho] = \frac{1}{m-1} \int \rho^m dx$ is convex along the curve if and only if $m \geq 2$.
3. This curve is Lipschitz in 2-Wasserstein metric.

Construction of the interpolation curve

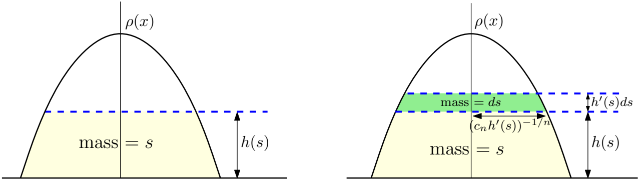
- Suppose ρ_0, ρ_1 are two radially decreasing step functions having N horizontal layers with mass $1/N$ in each layer.
- ρ_t is constructed by deforming each layer so that its height changes linearly, and meanwhile adjust the width so that the mass in each layer remains constant.



- Note that ρ_t is neither the linear interpolation between ρ_0 and ρ_1, nor the geodesic in 2-Wasserstein metric.
- For two radially decreasing function, the interpolation can be seen as a $N \to \infty$ limit of the step-function case.

Construction of the interpolation curve

- For a radially decreasing function ρ with mass 1, define its "height function with respect to mass" $h(s)$ as the left figure:



- $h : [0,1] \to [0, \|\rho\|_\infty]$ is increasing and convex in s. Also, ρ can be uniquely recovered from h (see the right figure):

$$\rho(x) = \int_0^1 1_{B(0,(c_d h'(s))^{-1/d})}(x) h'(s) ds$$

- Let h_0, h_1 be the height function for ρ_0, ρ_1. For $t \in (0,1)$, let

$$h_t(s) = (1-t)h_0(s) + th_1(s),$$

and let ρ_t be determined by the height function h_t.

Convexity of energy

- For the entropy, an explicit computation gives

$$S[\rho] = \int_{\mathbb{R}^d} \frac{1}{m-1}\rho^m dx$$
$$= \int_0^{\max\rho} \frac{m}{m-1} h^{m-1}|\{\rho > h\}|dh$$
$$= \int_0^1 \frac{m}{m-1} h(s)^{m-1} ds,$$

thus

$$\frac{d^2}{dt^2}S[\rho_t] = m(m-2)\int_0^1 (h_1 - h_0)^2 h_t(s)^{m-3} ds,$$

which is positive if and only if $m > 2$.

- The interaction energy $I[\rho] = \int \rho(\rho * W)dx$ is strictly convex along the curve for all attractive potential W, but the proof is more technical.

Non-uniqueness for $1 < m < 2$

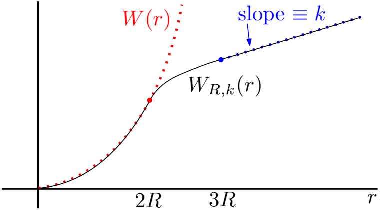
For all $m < 2$, our uniqueness proof fails. But is there really non-uniqueness in this regime?

Theorem (Delgadino–Yan–Y., '19)

Let $1 < m < 2$. Given any attractive kernel W with a stationary solution ρ_s, we can modify the tail of W (which remains attractive after the modification), such that it gives an infinite sequence of radially decreasing stationary solutions with the same mass.

- It shows that the uniqueness result for $m \geq 2$ is indeed sharp.
- Key step in the construction: starting from a stationary solution ρ_s, can we modify the tail of W so it leads to another stationary solution with the same mass, while ρ_s remains stationary?
- ρ_s is known to be compactly supported for $m > 1$; call its support $B(0, R)$. We will modify W outside $B(0, 2R)$. (so that ρ_s remains stationary.)

Modifying the tail



- Claim: If $k > 0$ is sufficiently small, then it leads to a different stationary solution from ρ_s.
- Reason: If $k = 0$, then W become an integrable attractive kernel. For such kernel, a heuristic scaling argument shows a sufficiently flat initial data should continue spreading for $1 \le m < 2$.

Scaling argument for integrable kernels

Assume that W is a integrable attracting kernel. As we replace ρ by $\rho_\lambda := \lambda^d \rho(\lambda x)$, the entropy and interaction energy scales as follows as $\lambda \to 0$:

$$S[\rho_\lambda] = \lambda^{(m-1)d} S[\rho] = \frac{\lambda^{(m-1)d}}{m-1} \int \rho^m dx,$$

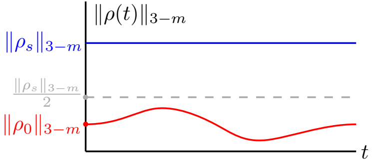
$$I[\rho_\lambda] \to \frac{\lambda^d}{2} \|W\|_{L^1} \int \rho^2 dx + o(\lambda^d).$$

Thus we formally expect the following:

- $m = 2$ (critical power): here both terms scale the same as $\lambda \to 0$.
- $1 \leq m < 2$: $E[\rho_\lambda] > 0$ for sufficiently small $\lambda > 0$.
(i.e. It is energy favorable for a sufficiently flat initial data to spread more.)
- $m > 2$: $E[\rho_\lambda] < 0$ for sufficiently small $\lambda > 0$.

Leading to non-uniqueness

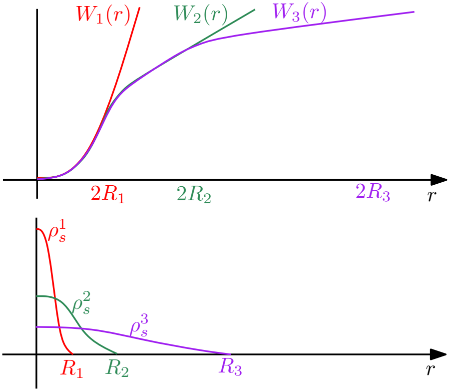
- Let $1 < m < 2$. To rigorously justify the heuristics, we use a standard energy estimate to track the evolution of L^{3-m} norm of a solution.
- We show if $0 < k \ll 1$ and $\|\rho_0\|_{3-m} \le \frac{\|\rho_s\|_{3-m}}{2}$, then $\|\rho(t)\|_{3-m}$ is bounded by $\frac{\|\rho_s\|_{3-m}}{2}$ for all times, so $\rho(t)$ can never return to ρ_s.



- But $\{\rho(t)\}_{t>0}$ must remain tight, since $W(r) \sim kr$ for $r \gg 1$, implying the first moment of $\rho(t)$ is uniformly bounded in time.
- Uniform-in-time L^{3-m} bounds + tightness + energy dissipation $\Rightarrow$ existence of a new stationary solution.

Infinite sequence of stationary solutions

- Finally, an iterative procedure allows us to construct a kernel with an infinite number of stationary solutions (all with the same mass, and radially decreasing).



Open questions

Questions

For a given mass, are stationary solutions unique (up to a translation) when $m = 1$?

- They are still radially decreasing, but both our uniqueness and non-uniqueness proof fails in the $m = 1$ case.

Questions

When $m > 2$, does the dynamical solution converges to the unique stationary solution with the same mass and center of mass as the initial data?

- Difficulty: need to show mass can't escape to infinity.

Thank you for your attention!