Existence and asymptotic behavior of positive solutions for a class of quasilinear Schrödinger equations with parameters

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We discuss the following quasilinear elliptic equation

$$-\Delta u + V(x)u - \gamma[\Delta(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}u = \lambda k(x,u), \qquad (0.1)$$

where $x \in \mathbb{R}^N, N \geq 3$, k is a nonlinear function including critical growth and subcritical perturbation;

γ and λ are parameters.

the potential $V(x) : \mathbb{R}^N \to \mathbb{R}$ is positive.

1. Motivation
2. Main Results
3. Outline of Proof

1 Motivation

1) Consider quasilinear Schodinger equation

$$i\partial_t z = -\Delta z + W(x)z - k(x,z) - \Delta l(|z|^2)l'(|z|^2)z \tag{1.1}$$

Set $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, (1.1) can be reduce to the corresponding equation of elliptic type:

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = k(x,u). \tag{1.2}$$

If we take

$$g^2(u) = 1 + \frac{((l(u^2))')^2}{2},$$

then (1.2) turns into

$$-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = k(x,u), \tag{1.3}$$

If we set $g^2(u) = 1 + 2u^2$, i.e., $l(s) = s$, we get the superfluid film equation in plasma physics:

$$-\Delta u + V(x)u - \Delta(u^2)u = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

If we set $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, i.e., $l(s) = (1+s)^{\frac{1}{2}}$, we get the equation:

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}u = k(x,u), \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter.

Consider the quasilinear problem

$g^2(u) = 1 + 2u^2, \quad (l(s) = s)$:

$$\begin{cases} -\Delta u - \Delta(u^2)u + V(x)u = k(x,u), & x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

The variational functional corresponding to (1.6) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V u^2 dx - \int_{\mathbb{R}^N} K dx,$$

which not well defined in $H^1(\mathbb{R}^N)$.

i). Perturbation method.

J. Liu, X. Liu and Z. Wang (Proc. AMS,2013; JDE,2013; CPDE,2014)

X. Wu and K. Wu (Nonlinearity,2014)

L. Jeanjean, T. LUO and Z. Wang (JDE,2015)

ii). Change of variables.

M. Colin and L. Jeanjean (Nonlinear Anal. TMA,2004)

J. Liu, Y. Wang and Z. Wang (JDE,2003)

iii). Constrained minimization or Nehari method.

M. Poppenberg, K. Schmitt and Z. Wang (CVPDE,2002)

J. Liu and Z. Wang (Proc. AMS,2003)

D. Ruiz and G. Siciliano (Nonlinearity, 2010)

J. Liu, Y. Wang and Z. Wang (CPDE,2004)

Y. Deng, S. Peng and J. Wang (CMS,2011; JMP,2013)

Consider following problem:

$$\begin{cases} -\Delta u - \Delta(u^2)u + V(x)u = |u|^{p-2}u, & x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

i) Exist positive solution if $4 < p < 2 \cdot 2^*$;

ii) No positive solution if $p \geq 2 \cdot 2^*$.

- Poppenberg et al.(see Calc. Var. PDE (2002)).
- Liu (J. Liu Wang and Wang, Comm. PDE 2004)
- Liu (J. Liu Wang and Wang, JDE 2003)

Critical problem with subcritical perturbation:

$$\begin{cases} -\Delta u - \Delta(u^2)u + V(x)u = |u|^{22^*-2}u + \lambda|u|^{p-2}u, \\ u \to 0, \quad \text{as } |x| \to \infty, \quad x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

Moameni (JDE (2006)) considered the related singularly perturbed problem and obtained a positive radial solution in the radially symmetric case.

João Marcos et al, (JDE(2010)): An existence result of positive solutions was obtained by via Mountain-Pass lemma.

Liu, Liu and Wang, (JDE (2013)): An existence result of positive solutions was obtained by via perturbation method.

Recently, we discussed the sign-change solutions for critical problem (1.6)

The following theorem is established for the existence of k-node solutions of problem (1.6), (JMP, 2013).

Theorem 1.1 *Assume that $V(x)$ satisfies (V_1). Then for all $\lambda > 0$, problem (1.6) exists at least one pair of k-node solutions if one of the following hold:*

i) $N \geq 6, \qquad 4 < q < 22^*$ *and* $\lambda > 0$,

ii) $3 \leq N < 6$, $\frac{2(N+2)}{N-2} < q < 22^*$ *and* $\lambda > 0$;

iii) $3 \leq N < 6$, $4 < q \leq \frac{2(N+2)}{N-2}$ *and* $\lambda > 0$ *large.*

Consider the general quasilinear elliptic problem ($g(u)$ is a general function)

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(u), \tag{1.7}$$

The problem with subcritical growth:

1) Existence of positive solution is obtained by Shen and Wang.

2) Existence of k-node solutions is obtained by Deng, Peng and Wang.

The problem with critical growth:

We find that the critical exponents for quasilinear problem (1.7) with general $g(s)$ are $\alpha 2^*$ if $\lim_{t\to+\infty} \frac{g(t)}{t^{\alpha-1}} = \beta > 0$ for some $\alpha \geq 1$. Consider

$$\begin{aligned} &-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + a(x)u \\ &= |u|^{\alpha 2^*-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.8}$$

where $2^* = \frac{2N}{N-2}$.

Theorem 1.2 *(Deng, Peng and Yan, JDE 2016)* *Problem (1.8), exists at least one positive solution if*

either $N \geq \max\left\{2 + \frac{4\alpha}{p-(\alpha+\gamma^+)},\ 4\right\}$ *and* $p > 2\alpha$,

or $N = 3$ *and* $p > 5\alpha + \gamma^+$, *where* $\gamma^+ = \max\{\gamma,\ 0\}$.

Remark 1): The case for $p \leq 2\alpha$:

$$-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + u = |u|^{\alpha 2^*-2}u + \lambda|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.9}$$

Theorem 1.3 *(Y. Deng, W. Huang and S. Zhang) Problem* (1.9) *exists a positive ground state solution if one of the following assumptions hold:*

(1) $p > \frac{\alpha(N+2)}{N-2} + \gamma^+$ *for* $3 \leq N < 6$ *and* $\lambda > 0$*;*

(2) $p > \max\{\frac{\alpha(N+2)}{N-2} + \gamma^+, 2\alpha\}$ *for* $N \geq 6$ *and* $\lambda > 0$*;*

(3) $2 < p < \alpha 2^*$ *for* $N \geq 3$ *and* $\lambda > 0$ *sufficiently large.*

Remark 2): $\alpha 2^* = \frac{2\alpha N}{N-2}$ behaves like a critical exponent since for

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + a(x)u = |u|^{q-2}u,$$

we can deduce the nonexistence of the positive solution in $H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx < \infty$ under the assumption (g_1) if $q \geq \alpha 2^*$ and $x \cdot \nabla a(x) \geq 0$ in $\mathbb{R}^N$.

The assumption for $g(t)$:

(g_1) $g \in C^1(\mathbb{R})$ is an even positive function and $g'(t) \geq 0$ for all $t \geq 0$, $g(0) = 1$. Moreover, there exist some constants $\alpha \geq 1$, $\beta > 0$ and $\gamma \in (-\infty, \alpha)$ such that

$$\begin{aligned} &g(t) = \beta t^{\alpha-1} + O(t^{\gamma-1}) \ \text{ as } \ t \to +\infty, \\ &(\alpha - 1) g(t) \geq g'(t) t, \ \ \forall\, t \geq 0; \end{aligned} \tag{1.10}$$

Note that the the second inequality of (1.10) on the assumption (g_1) is not satisfied if we take $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$ (i.e., $l(s) = (1+s)^{\frac{1}{2}}$) which correspond to problem (1.5). Since

$$\lim_{u \to +\infty} g(u) = \lim_{u \to +\infty} \sqrt{1 + \frac{u^2}{2(1+u^2)}} = \sqrt{3/2},$$

we obtain, in this case, that $\alpha = 1$ and $\beta = \sqrt{3/2}$ and hence the corresponding critical exponent is $\alpha 2^* = 2^*$.

Consider $g^2(t) = 1 + \frac{t^2}{2(1+t^2)}$, $\quad (l(s) = (1+s)^{\frac{1}{2}}$) Take $k(x,u) = |u|^{p-2}u$

$$-\Delta u - [\Delta(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}u + V(x)u = |u|^{p-2}u, \tag{1.11}$$

The variational functional corresponding to (1.11) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1 + \frac{u^2}{2(1+u^2)})|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} Vu^2 dx - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx,$$

which is well defined in $H^1(\mathbb{R}^N)$, but not smooth.

i) Exist positive solution if $12 - 4\sqrt{6} < p < 2^*$ $\ i.e.\ $ $(2.2 < p < 2^*)$;

ii) No positive solutions if $p \geq (6 - 2\sqrt{6})2^* \approx 1.1 \times 2^*$.

Questions:

1) 2^*-critical exponent?

2) Existence for $p \in (2, 12 - 4\sqrt{6})$?

3) Existence for $p \in (2^*, (6 - 2\sqrt{6})2^*)$?

To consider Eq.(1.11) when $p \in (2, 12 - 4\sqrt{6})$, we assume that the potential $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies the following conditions:

(V_1) $0 < V_0 \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x)$ for all $x \in \mathbb{R}^N$;

(V_2) there exists a function $\phi \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$|x \nabla V(x)| \leq \phi^2(x), \quad \forall x \in \mathbb{R}^N.$$

Theorem (Y. Deng and W. Huang):
Assume (V_1) and (V_2) hold. Then problem (1.11) exists at least one positive solution if either $p \in (2, 2^*)$ for $N \geq 4$ or $p \in (4, 6)$ for $N = 3$.

Question: Assumption (V_2)? Necessary?

2 Main results

We discuss the following quasilinear elliptic equation

$$-\Delta u + V(x)u - \gamma[\Delta(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}u = \lambda|u|^{p-2}u, \qquad (2.1)$$

where $x \in \mathbb{R}^N, N \geq 3, p > 2$;

γ and λ are parameters, the potential $V(x) : \mathbb{R}^N \to \mathbb{R}$ is positive.

Theorem 2.1 *Assume that* (V_1) *and* $p > 2$, $N \geq 3$. *Then, the following statements hold:*

(1) *for all* $\lambda > 0$ *and* $p \in (2, 2^*)$, *equation (2.1) has a positive classical solution if* $\gamma \in (0, \gamma^*)$, *where*

$$\gamma^* = \begin{cases} \frac{16(p-2)}{(p-4)^2}, & \text{if } p < 4, \\ +\infty, & \text{if } p \geq 4 \end{cases};$$

(2) *for all* $\gamma > 0$ *and* $p \in (2, 2^*)$, *equation (2.1) has a positive classical solution if* $\lambda \in (\lambda^*, +\infty)$, *where*

$$\lambda^* = (p-2)^{\frac{2-p}{2}} \left(\frac{2^* - p + 2}{2}\right)^{\frac{2(2^*-p+2)(p-2)}{(2^*-p)^2}} 2^{\frac{7\cdot 2^*-2-6p}{2(2^*-p)}} S^{-\frac{(2^*-2)(p-2)}{2(2^*-p)}} (2+\gamma)^{\frac{p(2^*-2)}{2(2^*-p)}} \gamma^{\frac{p-2}{2}}$$

and S *is the best Sobolev constant of inequality* $S\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2$, $u \in D^{1,2}(\mathbb{R}^N)$.

(3) *for all* $\gamma, \lambda > 0$, *there exists a constant* $p^* \in [2^*, \min\{\frac{9+2\gamma}{8+2\gamma}, \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}\}2^*)$ *such that equation (2.1) has no positive solution if* $p \in [p^*, +\infty)$ *and* $\nabla V(x) \cdot x \geq 0$ *in* $\mathbb{R}^N$.

Theorem 2.2 *Suppose* $V(x) = \mu = constant > 0$, $p \in (2, 2^*)$, *then the corresponding solution* $u_{\gamma,\lambda}$ *of equation (2.1) obtained in Theorem 2.1 is spherically symmetric and monotone decreasing with respect to* $r = |x|$. *Passing to a subsequence if necessary, we have*

$$u_{\gamma,\lambda} \to u_\lambda \;\; in \;\; H^2(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \;\; as \;\; \gamma \to 0^+,$$

where u_λ *is the ground state of semilinear problem*

$$-\Delta u + \mu u = \lambda |u|^{p-2} u, \;\; u \in H^1(\mathbb{R}^N). \tag{2.2}$$

3 Sketch of Proof

The energy functional corresponding to (2.1) is

$$\begin{aligned}\widetilde{I}_{\gamma,\lambda}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(1+\frac{\gamma u^2}{2(1+u^2)}\right)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx\\ &\quad -\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx.\end{aligned}$$

Denote $\widetilde{g}_\gamma(t)=\sqrt{1+\frac{\gamma t^2}{2(1+t^2)}}$, we have

$$\begin{aligned}\widetilde{I}_{\gamma,\lambda}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\widetilde{g}_\gamma(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx\\ &\quad -\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx.\end{aligned}$$

$\widetilde{g}_\gamma(t)$ does not satisfy the assumption: $(\alpha-1)\widetilde{g}(t)\ge\widetilde{g}'(t)t\ge 0,\ \ \forall\ t\ge 0$ since $\alpha=1$.

Step 1: Find a new function $g_\gamma(t)$ such that $g_\gamma(t) = \widetilde{g}_\gamma(t)$ if $t \in [0, \delta_\gamma)$ and $g_\gamma(t)$ satisfy $\frac{p-2}{2} g(t) \geq g'(t)t \geq 0, \ \ \forall\, t > 0$;

$$g_\gamma(t) = \sqrt{\frac{1}{2}\left(1 + \frac{\gamma t^2}{1+t^2}\right)\eta(t) + \frac{1}{2}},$$

where $\eta(t)$ is a spatial function satisfying either the following (η_1) or (η_2):

(η_1) $\eta(t) \equiv 1$, for all $t \in \mathbb{R}$;

(η_2) $\eta(t) \in C_0^\infty(\mathbb{R}, [0,1])$ is a cut-off function satisfying

$$\eta(t) \begin{cases} = \eta(-t), & \text{if } t \leq 0, \\ = 1, & \text{if } 0 \leq t \leq \delta_\gamma := \frac{1}{4}\sqrt{\frac{p-2}{\gamma}}, \\ \in (0,1), & \text{if } \frac{1}{4}\sqrt{\frac{p-2}{\gamma}} < t < \frac{1}{2}\sqrt{\frac{p-2}{\gamma}}, \\ = 0, & \text{if } t \geq \frac{1}{2}\sqrt{\frac{p-2}{\gamma}}, \end{cases} \tag{3.1}$$

where $p \in (2, 2^*)$. Moreover, it also satisfies

$$-\sigma\sqrt{\eta(t)} \leq \eta'(t)t \leq 0, \quad \text{for all } t \in \mathbb{R}, \tag{3.2}$$

where σ is a positive constant independent of γ.

Step 2: By changing variables, reduce the problem to Semilinear one;

The energy functional corresponding to g_γ is

$$\begin{aligned} I_{\gamma,\lambda}(u) &= \tfrac{1}{2}\int_{\mathbb{R}^N} g_\gamma(u)|\nabla u|^2 dx + \tfrac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx \\ &-\tfrac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p dx. \end{aligned}$$

Introduce a change of known variables $v = G_\gamma(u) = \displaystyle\int_0^u g_\gamma(t)dt$,

Then $I_{\gamma,\lambda}(u)$ can be rewritten by

$$\begin{aligned} J_{\gamma,\lambda}(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|G_\gamma^{-1}(v)|^2\right) dx \\ &-\frac{\lambda}{p}\int_{\mathbb{R}^N} |G_\gamma^{-1}(v)|^p dx. \end{aligned}$$

All nontrivial critical points of $J_{\gamma,\lambda}$ are the nontrivial solutions of

$$-\Delta v + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} - \frac{\lambda|G_\gamma^{-1}(v)|^{p-2}G^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} = 0. \tag{3.3}$$

Step 3: To find the positive solution $v_{\gamma,\lambda}$ for semilinear problem (3.3);

Step 4: To estimate $v_{\gamma,\lambda}$;

$$||v_{\gamma,\lambda}||_\infty \leq \left(\frac{2^* - p + 2}{2}\right)^{\frac{2(2^*-p+2)}{(2^*-p)^2}} 2^{\frac{2\cdot 2^*-2-p}{2(2^*-p)}} S^{-\frac{2^*-2}{2(2^*-p)}} \left(\frac{1}{2+\gamma}\right)^{\frac{p(2^*-2)}{2(2-p)(2^*-p)}} \lambda^{\frac{1}{2-p}}.$$

Step 5: To prove Theorem 1;

Proof of Theorem 2.1–(1): For all $\gamma > 0$, if $p \in (2, 2^*)$ and $\gamma \in (0, \gamma^*)$, we take $\eta(t)$ satisfying (η_1). In this case, $\tilde{g}_\gamma(t) = g_\gamma(t)$. It follows that $u_{\gamma,\lambda} = G_\gamma^{-1}(v_{\gamma,\lambda}) > 0$ is a solution of (2.1).

Proof of Theorem 2.1–(2): From Step 4, for any $\gamma > 0$, we set $K = \left(\frac{2^*-p+2}{2}\right)^{\frac{2(2^*-p+2)}{(2^*-p)^2}} 2^{\frac{2\cdot 2^*-2-p}{2(2^*-p)}} S^{-\frac{2^*-2}{2(2^*-p)}} \left(\frac{1}{2+\gamma}\right)^{\frac{p(2^*-2)}{2(2-p)(2^*-p)}}$ and choose $\lambda^* = d\gamma^{\frac{p-2}{2}}$ with $d = (\frac{\sqrt{p-2}}{4\sqrt{2}K})^{2-p}$ such that

$$||u_{\gamma,\lambda}||_\infty = ||G_\gamma^{-1}(v_{\gamma,\lambda})||_\infty$$

$$\leq \sqrt{2}||v_{\gamma,\lambda}||_\infty \leq \sqrt{2}K\lambda^{\frac{1}{2-p}} \leq \frac{1}{4}\sqrt{\frac{p-2}{\gamma}}, \quad \forall\lambda \in (\lambda^*, +\infty).$$

In this case, we take $\eta(t)$ satisfying (η_2). It follows from above estimate that $\tilde{g}_\gamma(t) = g_\gamma(t)$ if $\lambda \in (\lambda^*,\ +\infty)$ and hence $u_{\gamma,\lambda} = G_\gamma^{-1}(v_{\gamma,\lambda}) > 0$ is a solution of (2.1).

Proof of Theorem 2.1–(3): We are going to find a constant

$$p^* \in [2^*, \min\{\frac{9+2\gamma}{8+2\gamma}, \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}\}2^*)$$

such that problem (2.1) has no positive solution $u \in H^1(\mathbb{R}^N)$ for $p \geq p^*$ if $x \cdot \nabla V(x) \geq 0$ in $\mathbb{R}^N$. It suffices to prove that problem (3.3) has no positive solution.

Suppose by contrary that $v \in H^1(\mathbb{R}^N)$ is a positive solution of (3.3), it follows from the Pohozaev identity that

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))|G_\gamma^{-1}(v)|^2dx = \int_{\mathbb{R}^N} K(G_\gamma^{-1}(v))dx \\
&=: \int_{\{x\in\mathbb{R}^N:\, 0\le u<\frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx + \int_{\{x\in\mathbb{R}^N:\, u\ge\frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx,
\end{aligned}
\tag{3.4}
$$

where $u = G_\gamma^{-1}(v)$ and

$$
K(u) = \frac{(N-2)\lambda}{2}\frac{G_\gamma(u)u^{p-1}}{g_\gamma(u)} - \frac{N\lambda}{p}u^p + \frac{N}{2}u^2 - \frac{N-2}{2}\frac{G_\gamma(u)u}{g_\gamma(u)}.
$$

The assumption $x\cdot\nabla V(x) \ge 0$ implies that

$$
-\frac{1}{2}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))|G_\gamma^{-1}(v)|^2dx < 0.
$$

Therefore, to complete the proof of our theorem 2.1–(3), it suffices to verify that the right hand side of (3.4) is nonnegative.

Using Lemma 2.1, we get $K(u) > 0$ if $p \geq \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}2^* > 2^*$. Noting that $\frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma} \to 1$ as $\gamma \to 0$. Hence, we only need to consider the case $p \in [2^*, \frac{2\gamma+4-2\sqrt{4+2\gamma}}{\gamma}2^*)$.

Noting that

$$K(u) \geq \frac{(N-2)\lambda}{2}\frac{G_\gamma(u)u^{p-1}}{g_\gamma(u)} - \frac{N\lambda}{2^*}u^p + \frac{N}{2}u^2 - \frac{N-2}{2}\frac{G_\gamma(u)u}{g_\gamma(u)}$$
$$= \frac{N-2}{2}\frac{u}{g_\gamma(u)}\big(ug_\gamma(u) - G_\gamma(u)\big)\big(1-\lambda u^{p-2}\big) + u^2, \tag{3.5}$$

we see

$$\int_{\{x\in\mathbb{R}^N:\,0\leq u<\frac{1}{\lambda^{\frac{1}{p-2}}}\}} K(u)dx > 0. \tag{3.6}$$

Observing (3.5), we can choose $\bar{t} > \frac{1}{\lambda^{\frac{1}{p-2}}}$ (which can be independent of p) such that $K(t) \geq 0, \ \forall\, t \in [\frac{1}{\lambda^{\frac{1}{p-2}}}, \bar{t}]$. Now, by direct calculation, we see

$$\begin{aligned}\frac{tg'_\gamma(t)}{g_\gamma(t)} &= \frac{1}{2t^{-2} + (4+\gamma) + (2+\gamma)t^2} \\ &\leq \frac{1}{2\bar{t}^{-2} + (4+\gamma) + (2+\gamma)\bar{t}^2} =: \eta(\bar{t}) \leq \frac{1}{8+2\gamma}, \quad \forall\, t \geq \bar{t}.\end{aligned}$$

Hence, if we choose $p \geq (1+\eta(\bar{t}))2^* =: p^*$, we find

$$\begin{aligned}K(u) &= \frac{N\lambda u^{p-1}}{pg_\gamma(u)}\Big(\frac{p}{2^*}G_\gamma(u) - ug_\gamma(u)\Big) + \frac{N-2}{2}\big(ug_\gamma(u) - G_\gamma(u)\big) + u^2 \\ &> \frac{N\lambda u^{p-1}}{pg_\gamma(u)}\Big[(1+\eta(\bar{t}))G_\gamma(u) - ug_\gamma(u)\Big] \geq 0,\end{aligned}$$

which combined with (3.6) implies that the right hand side of (3.4) is positive.

Remark: Since we can not find the explicit form of $G_\gamma(t)$, it is difficult for us to give the exact value of $\bar{t}$, below which $K(u)$ in (3.5) is non-negative. However, we guess that $\bar{t}$ there should be $+\infty$, which implies that p^* is exactly 2^*, the critical exponent.

Thank you for your attention !