Some progress on Leray's Problem

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Leray's problem in Channels

Consider steady incompressible Navier-Stokes equations

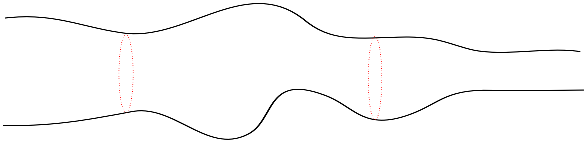
$$\begin{cases} \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \Delta \mathbf{u}, \\ \operatorname{div} \mathbf{u} = 0 \end{cases} \tag{1}$$

in a nozzle domain Ω which tends to a straight cylinder $\hat{\Omega}$ in the asymptotic ends.

Leray's Problem: Find a solution of the system (1) satisfying the no slip condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

and tending to Poiseuille flows with the flux Φ at far fields.



Poiseuille flow and Hagen-Poiseuille Flows

The Poiseuille flows in the straight cylinder $\Sigma \times \mathbb{R}$ has the form $\mathbf{v} = (0, 0, v^z(x, y))$ which satisfies

$$\begin{cases} \Delta u^z = Constant \quad \text{in } \Sigma, \\ u^z = 0 \quad \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma u^z(x, y)\, dxdy = \Phi. \end{cases}$$

Clearly, the Poiseuille flow is uniquely determined by Φ.
In the case that $\Sigma = B_1(0)$, the Poiseuille flow has the explicit form

$$\mathbf{u} = \bar{U}(r)\mathbf{e}_z := \frac{2\Phi}{\pi}(1 - r^2)\mathbf{e}_z \tag{2}$$

where $r = \sqrt{x^2 + y^2}$. This is also called Hagen-Poiseuille flow.

Progress and Problem

- Amick(1977): Existence of unique weak solution under a smallness assumption on $|\Phi|$.
- Ladyzhenskaya and Solonnikov(1980): Existence of a weak solution without any assumption on $|\Phi|$. However, there is no global estimate for $\mathbf{v}$.
- ...

Open Problem (Galdi's book): Existence and uniqueness of solutions with large flux Φ? Far field behavior?

Linearized Problem

Hence we consider the following problem

$$\begin{cases} -\Delta \mathbf{v} + \bar{U}(r)\partial_z \mathbf{v} + \mathbf{v} \cdot \nabla \left[\bar{U}(r)\mathbf{e}_z\right] + \nabla p = \mathbf{F}, \\ \mathrm{div}\ \mathbf{v} = 0, \\ \int_\Sigma \mathbf{v} \cdot \mathbf{n}\, dS = 0 \end{cases} \tag{3}$$

with boundary condition

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

Question: Given $\mathbf{F} \in L^2(\Omega)$, can we show the existence of unique solution $\mathbf{v} \in H^2(\Omega)$, with

$$\|\mathbf{v}\|_{H^2(\Omega)} \leq C\|\mathbf{F}\|_{L^2(\Omega)}?$$

Linear Structural Stability in the Axisymmetric Setting

Theorem 1 Given
$\mathbf{F} = F^r(r,z)\mathbf{e}_r + F^z(r,z)\mathbf{e}_z + F^\theta(r,z)\mathbf{e}_\theta \in L^2(\Omega)$. There exists a unique axisymmetric solution $\mathbf{v} \in H^2(\Omega)$ to the linearized problem, where the solution

$$\mathbf{v}(x,y,z) = \mathbf{v}(r,z) = v^r(r,z)\mathbf{e}_r + v^z(r,z)\mathbf{e}_z + v^\theta \mathbf{e}_\theta,$$

and satisfies

$$\|\mathbf{v}\|_{H^2(\Omega)} \leq C(1 + \Phi^{\frac{1}{4}})\|\mathbf{F}\|_{L^2(\Omega)} \tag{5}$$

and

$$\|\mathbf{v}\|_{H^{\frac{5}{3}}(\Omega)} \leq C\|\mathbf{F}\|_{L^2(\Omega)},$$

where C is independent of Φ.

Linear Dynamical Stability

- Linearized perturbation equations:

$$\begin{cases} s\mathbf{v} - \Delta\mathbf{v} + \bar{U}(r)\partial_z\mathbf{v} + \mathbf{v}\cdot\nabla(\bar{U}(r)\mathbf{e}_z) + \nabla p = \mathbf{F}, \\ \mathrm{div}\ \mathbf{v} = 0. \end{cases} \tag{6}$$

- Dirichlet boundary condition:

$$\mathbf{v}(x,y,z) = 0, \qquad when \quad r^2 = x^2 + y^2 = 1. \tag{7}$$

- Zero flux condition:

$$\int_{B_1(0)} \mathbf{v}\cdot\mathbf{n}\, dS = 0. \tag{8}$$

Problem: Does the spectral s of the problem (6)-(8) satisfy $\Re s < 0$?

Progress: When the flows are periodic in the axial direction, the spectrum analysis for the linearized problem has been obtained by Gong and Guo (2016) and Chen, Wei, and Zhang (2019) in axisymmetric and 3D setting, respectively.

Numerical Computations and Rigorous Analysis

The computation by Meseguer and Trefethen (JCP, 2003):

- The norm of the resolvent operator $\mathscr{R}(s)$ (as a map from L^2 to L^2) is maximized at $s = 0$
- $\|\mathscr{R}(0)\|_{L^2 \to L^2} \sim Re^2$

Further Remark on the Norm of the Resolvent

Note $\mathbf{v} = \mathscr{R}(0)\mathbf{F}$ if

$$\begin{cases} \frac{2(1-r^2)}{\pi}\partial_z \mathbf{v} + \mathbf{v}\cdot\nabla\left[\frac{2(1-r^2)}{\pi}\mathbf{e}_z\right] + \nabla p = \frac{1}{Re}\Delta\mathbf{v} + \mathbf{F}, \\ \mathrm{div}\mathbf{v} = 0, \end{cases} \tag{9}$$

It follows from Theorem 1 that one has

$$\|\mathbf{v}\|_{L^2} \leq C\cdot Re\|\mathbf{F}\|_{L^2}.$$

This implies at least in the axisymmetric setting,

$$\|\mathscr{R}(0)\|_{L^2\to L^2} \leq C\cdot Re.$$

Uniform Nonlinear Structural Stability

Theorem 2 Given
$\mathbf{F} = F^r(r,z)\mathbf{e}_r + F^z(r,z)\mathbf{e}_z + F^\theta(r,z)\mathbf{e}_\theta \in L^2(\Omega)$. There exists a positive constant ϵ_0, such that if

$$\|\mathbf{F}\|_{L^2(\Omega)} \leq \epsilon_0,$$

the Navier-Stokes equations with Dirichlet boundary condition has a unique axisymmetric solution $\mathbf{u} \in H^2(\Omega)$ satisfying

$$\|\mathbf{u} - \bar{U}(r)\mathbf{e}_z\|_{H^{\frac{5}{3}}(\Omega)} \leq C\|\mathbf{F}\|_{L^2(\Omega)}, \tag{10}$$

and

$$\|\mathbf{u} - \bar{U}(r)\mathbf{e}_z\|_{H^2(\Omega)} \leq C(1+\Phi^{\frac{1}{4}})\|\mathbf{F}\|_{L^2(\Omega)}, \tag{11}$$

where C is independent of Φ.

Remarks on Structural Stability

- There is no restriction on the size of flux $|\Phi|$.
- The Hagen-Poiseuille flow is uniformly structural stable with respect to the flux Φ for a given $\mathbf{F}$.

The Linearized System

The linearized system for the axially symmetric solutions for the Navier-Stokes system is

$$\begin{cases} \bar{U}(r)\dfrac{\partial u^r}{\partial z} + \dfrac{\partial P}{\partial r} - \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u^r}{\partial r}\right) + \dfrac{\partial^2 u^r}{\partial z^2} - \dfrac{u^r}{r^2}\right] = F^r\,, \\ u^r\dfrac{\partial \bar{U}(r)}{\partial r} + \bar{U}(r)\dfrac{\partial u^z}{\partial z} + \dfrac{\partial P}{\partial z} - \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u^z}{\partial r}\right) + \dfrac{\partial^2 u^z}{\partial z^2}\right] = F^z\,, \\ \partial_r u^r + \partial_z u^z + \dfrac{u^r}{r} = 0\,, \end{cases} \tag{12}$$

and

$$\bar{U}(r)\cdot\partial_z u^\theta - \Delta u^\theta = F^\theta. \tag{13}$$

Observation: The system for swirl component velocity and the other two components (radially and axially components) are decoupled.

Vorticity and Stream Functions

Let $\omega^\theta = \partial_r u^z - \partial_z u^r$. Then ω^θ satisfies

$$\bar{U}(r)\partial_z\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^\theta + \frac{\omega^\theta}{r^2} = f, \tag{14}$$

where $f = \partial_r F^z - \partial_z F^r$. Introduce the stream function $\psi(r,z)$ satisfying

$$\partial_z(r\psi) = -ru^r \qquad \partial_r(r\psi) = ru^z.$$

$$\omega^\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\psi)\right) + \partial_z^2\psi.$$

ψ satisfies the following equation

$$\bar{U}(r)\partial_z(\mathcal{L} + \partial_z^2)\psi - (\mathcal{L} + \partial_z^2)^2\psi = f, \tag{15}$$

where

$$\mathcal{L} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cdot)\right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}.$$

Fourier Transform and Boundary Conditions

Taking the Fourier transform in z variable yields

$$i\xi \bar{U}(r)(\mathcal{L}-\xi^2)\hat{\psi} - (\mathcal{L}-\xi^2)^2\hat{\psi} = \hat{f}. \tag{16}$$

Note that the boundary conditions for u^r and u^z are as follows

$$\begin{cases} u^r(1,z) = u^z(1,z) = 0, \\ \displaystyle\int_0^1 ru^z(r,z)\,dr = 0. \end{cases} \tag{17}$$

When we are working with classical solutions, we need the following compatibility condition (by Liu and Wang)

$$\psi(0,z) = (\mathcal{L}+\partial_z^2)\psi(0,z) = 0.$$

So the boundary conditions for $\hat{\psi}$ are as follows.

$$\begin{cases} \hat{\psi}(0) = \hat{\psi}(1) = \hat{\psi}'(1) = 0, \\ \mathcal{L}\hat{\psi}(0) = 0. \end{cases} \tag{18}$$

The First Estimate and Hardy Inequality-I

Multiplying the both sides of the equation with $\bar{\psi} r$ and integrating over the domain yield

$$\int_0^1 |\mathcal{L}\hat{\psi}|^2 r\,dr + 2\xi^2 \int_0^1 \left|\frac{d}{dr}(r\hat{\psi})\right|^2 \frac{1}{r}\,dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r\,dr$$
$$= -\Re \int_0^1 \hat{f}\overline{\hat{\psi}} r\,dr - \frac{4\Phi}{\pi}\xi\Im \int_0^1 \left[\frac{d}{dr}(r\hat{\psi})r\overline{\hat{\psi}}\right]\,dr,$$

and

$$\xi \int_0^1 \frac{\bar{U}(r)}{r}\left|\frac{d}{dr}(r\hat{\psi})\right|^2\,dr + \xi^3 \int_0^1 \bar{U}(r)|\hat{\psi}|^2 r\,dr$$
$$= -\Im \int_0^1 \hat{f}\overline{\hat{\psi}} r\,dr.$$

<u>Lemma 1</u> Let $g \in C^1([0,1])$ satisfy $g(0) = 0$, one has

$$\int_0^1 |g(r)|^2 r\,dr \le C \int_0^1 \left|\frac{d(rg)}{dr}\right|^2 \frac{(1-r^2)}{r}\,dr. \qquad (19)$$

The first Estimate and Hardy Inequality-II

Applying Lemma 1 to the imaginary part,

$$\Phi^2\xi^2\int_0^1 |\hat{\psi}|^2 r\,dr \le C\int_0^1 |\hat{f}|^2 r\,dr. \tag{20}$$

Hence one has

$$\begin{aligned}\left|\frac{4\Phi}{\pi}\xi\int_0^1 \frac{d}{dr}(r\hat{\psi})r\overline{\hat{\psi}}\,dr\right| \le & \frac{1}{4}\int_0^1 \left|\frac{d}{dr}(r\hat{\psi})\right|^2 \frac{1}{r}\,dr + C\Phi^2\xi^2\int_0^1 |\hat{\psi}|^2 r\,dr \\ \le & \frac{1}{4}\int_0^1 |\mathcal{L}\hat{\psi}|^2 r\,dr + C\int_0^1 |\hat{f}|^2 r\,dr.\end{aligned}$$

A Priori Estimate and Existence

Basic a priori estimate:

$$\xi^4 \int_0^1 \left|\hat{\psi}\right|^2 r\,dr + \xi^2 \int_0^1 \left|\frac{d}{dr}(r\hat{\psi})\right|^2 \frac{1}{r}\,dr + \int_0^1 |\mathcal{L}^2\hat{\psi}|^2 r\,dr$$
$$\leq C \int_0^1 |\hat{f}|^2 r\,dr.$$

High order a priori estimate:

$$\xi^4 \int_0^1 \left|\mathcal{L}\hat{\psi}\right|^2 r\,dr + \xi^6 \int_0^1 \left|\frac{d}{dr}(r\hat{\psi})\right|^2 \frac{1}{r}\,dr + \xi^8 \int_0^1 \left|\hat{\psi}\right|^2 r\,dr$$
$$\leq C(1+\Phi) \int_0^1 |\hat{f}|^2 r\,dr$$

and

$$\int_0^1 |\mathcal{L}^2\hat{\psi}|^2 r\,dr \leq C(1+\Phi^2) \int_0^1 |\hat{f}|^2 r\,dr.$$

Estimate for the Case with Small Flux

A more careful estimate shows

$$\int_{-\infty}^{+\infty}\int_0^1 \left\{ \left(|\mathcal{L}\hat{\psi}|^2 + \xi^4 |\hat{\psi}|^2 \right) r + \xi^2 \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 \frac{1}{r} \right\} dr d\xi$$
$$\leq C(1+\Phi^2)\|\mathbf{F}^*\|_{L^2(\Omega)}^2,$$

where $\mathbf{F}^* = F^r\mathbf{e}_r + F^z\mathbf{e}_z$. Note that $\boldsymbol{\omega}^\theta = (\mathcal{L} + \partial_z^2)\psi\mathbf{e}_\theta$, one has

$$\|\mathbf{v}^*\|_{H^1(\Omega)} \leq C\|\nabla\mathbf{v}^*\|_{L^2(\Omega)} = C\|\boldsymbol{\omega}^\theta\|_{L^2(\Omega)} \leq C(1+\Phi)\|\mathbf{F}^*\|_{L^2(\Omega)},$$

where $\mathbf{v}^* = v^r\mathbf{e}_r + v^z\mathbf{e}_z$. Applying the regularity theory for Stokes equations gives

$$\begin{aligned}\|\mathbf{v}^*\|_{H^2(\Omega)} &\leq C\|\mathbf{F}^*\|_{L^2(\Omega)} + \Phi\|\partial_z\mathbf{v}^*\|_{L^2(\Omega)} + \Phi\|v^r\|_{L^2(\Omega)} + C\|\mathbf{v}^*\|_{H^1(\Omega)} \\ &\leq C(1+\Phi^2)\|\mathbf{F}^*\|_{L^2(\Omega)}.\end{aligned}$$

Case with Large Flux and Low Frequency ($|\xi| \le \frac{1}{\epsilon_1 \Phi}$)

The energy estimate gives

$$\int_0^1 |\mathcal{L}\hat{\psi}|^2 r\, dr + 2\xi^2 \int_0^1 \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 \frac{1}{r}\, dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r\, dr \\ \le C(\epsilon_1) \int_0^1 |\widehat{\mathbf{F}^*}|^2 r\, dr. \tag{21}$$

Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| \le \dfrac{1}{\epsilon_1 \Phi}, \\ 0, & \textit{otherwise}, \end{cases}$$

and ψ_{low} be the function such that $\widehat{\psi_{low}} = \chi_1(\xi)\hat{\psi}$. Define

$$v_{low}^r = \partial_z \psi_{low}, \qquad v_{low}^z = -\frac{\partial_r(r\psi_{low})}{r}, \qquad \mathbf{v}_{low}^* = v_{low}^r \mathbf{e}_r + v_{low}^z \mathbf{e}_z.$$

$$\|\mathbf{v}_{low}^*\|_{H^2(\Omega)} \le C\|\mathbf{F}_{low}^*\|_{L^2(\Omega)}. \tag{22}$$

Case with Large Flux and high frequency ($|\xi| \geq \epsilon_1\sqrt{\Phi}$)-I

The energy estimate yields

$$\begin{aligned}&\int_0^1 |\mathcal{L}\hat{\psi}|^2 r\,dr + \xi^2 \int_0^1 \left|\frac{d}{dr}(r\hat{\psi})\right|^2 \frac{1}{r}\,dr + \xi^4 \int_0^1 |\hat{\psi}|^2 r\,dr \\ \leq &C|\xi|^{-2} \int_0^1 |\widehat{\mathbf{F}^*}|^2 r\,dr,\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\Phi|\xi| \int_0^1 \frac{1-r^2}{r} \left|\frac{d}{dr}(r\hat{\psi})\right|^2 dr + \Phi|\xi|^3 \int_0^1 (1-r^2)|\hat{\psi}|^2 r\,dr \\ \leq &C|\xi|^{-2} \int_0^1 |\widehat{\mathbf{F}^*}|^2 r\,dr.\end{aligned} \tag{24}$$

Thus one has

$$\|\mathbf{v}^*_{high}\|_{H^1(\Omega)} \leq C\|\boldsymbol{\omega}^\theta_{high}\|_{L^2(\Omega)} \leq C\Phi^{-\frac{1}{2}}\|\mathbf{F}^*_{high}\|_{L^2(\Omega)},$$

Case with Large Flux and High Frequency ($|\xi| \geq \epsilon_1\sqrt{\Phi}$)-II

where

$$\mathbf{v}^*_{high} = v^r_{high}\mathbf{e}_r + v^z_{high}\mathbf{e}_z \quad \text{with } v^r_{high} = \partial_z\psi_{high}, \quad v^z_{high} = -\frac{\partial_r(r\psi_{high})}{r}$$

and $\widehat{\psi_{high}} = \chi_2(\xi)\hat{\psi}$ with

$$\chi_2(\xi) = \begin{cases} 1, & |\xi| \geq \epsilon_1\sqrt{\Phi}, \\ 0, & \text{otherwise.} \end{cases}$$

The regularity estimate for the Stokes equations gives

$$\|\mathbf{v}^*_{high}\|_{H^2(\Omega)} \leq C(1 + \Phi^{\frac{1}{4}})\|\mathbf{F}^*_{high}\|_{L^2(\Omega)}, \tag{25}$$

Using interpolation gives

$$\|\mathbf{v}^*_{high}\|_{H^{\frac{5}{3}}(\Omega)} \leq C\|\mathbf{F}^*_{high}\|_{L^2(\Omega)},$$

Large Flux and Intermediate Frequency $(\frac{1}{\epsilon_1\Phi} \le |\xi| \le \epsilon_1\sqrt{\Phi})$

Inspired by the work by Gallagher, Higaki and Maekawa(2019), ψ can be decomposed into four parts,

$$\hat{\psi}(r) = \widehat{\psi_s}(r) + aI_1(|\xi|r) + b(\chi\widehat{\psi_{BL}} + \widehat{\psi_e}). \tag{26}$$

- $\widehat{\psi_s}$ is a solution to the following problem

$$\begin{cases} i\xi\bar{U}(r)(\mathcal{L}-\xi^2)\widehat{\psi_s} - (\mathcal{L}-\xi^2)^2\widehat{\psi_s} = \hat{f}, \\ \widehat{\psi_s}(0) = \widehat{\psi_s}(1) = \mathcal{L}\widehat{\psi_s}(0) = \mathcal{L}\widehat{\psi_s}(1) = 0. \end{cases} \tag{27}$$

- $\widehat{\psi_{BL}}$ is the boundary layer profile, which is the exact solution (exponentially decay away from $r = 1$) of the equation

$$\left(i\frac{\xi\Phi}{\pi}4(1-r) - \frac{d^2}{dr^2} + \xi^2\right)\left(\frac{d^2}{dr^2} - \xi^2\right)\widehat{\psi_{BL}} = 0. \tag{28}$$

One can represent $\widehat{\psi_{BL}}$ in terms of the Airy function.

The Error Term and Irrotational Flows

- $\widehat{\psi_e}$ is a remainder term, which satisfies the problem

$$\begin{cases} i\xi \bar{U}(r)(\mathcal{L}-\xi^2)(\chi\widehat{\psi_{BL}}+\widehat{\psi_e}) - (\mathcal{L}-\xi^2)^2(\chi\widehat{\psi_{BL}}+\widehat{\psi_e}) = 0, \\ \widehat{\psi_e}(0) = \widehat{\psi_e}(1) = \mathcal{L}\widehat{\psi_e}(0) = \mathcal{L}\widehat{\psi_e}(1) = 0, \end{cases}$$

  where χ is a smooth cut-off function satisfying

$$\chi(r) = 1 \;\text{ if }\; r \geq \frac{1}{2} \;\text{ and }\; \chi(r) = 0 \;\text{ if }\; r \leq \frac{1}{4}.$$

- $I_1(\rho)$ is the modified Bessel function of the first type, satisfying $I_1(z)$ is the modified Bessel function satisfying

$$\begin{cases} z^2\frac{d^2}{dz^2}I_1(z) + z\frac{d}{dz}I_1(z) - (z^2+1)I_1(z) = 0, \\ I_1(0) = 0, \; I_1(z) > 0 \text{ for } z > 0. \end{cases}$$

  This implies

$$(\mathcal{L}-\xi^2)I_1(|\xi|r) = 0.$$

Match the Boundary Conditions

The no-slip boundary conditions give

$$\begin{cases} aI_1(|\xi|) + b\widehat{\psi_{BL}}(1) = 0, \\ a|\xi|I_1'(|\xi|) + b\frac{d}{dr}\widehat{\psi_{BL}}(1) + b\frac{d}{dr}\widehat{\psi_e}(1) = -\frac{d}{dr}\widehat{\psi_s}(1). \end{cases} \tag{29}$$

One can get the following estimate

$$\|\mathbf{v}^*_{med}\|_{H^2(\Omega)} \leq C\|\mathbf{F}^*_{med}\|_{L^2(\Omega)}. \tag{30}$$

Estimate for the Swirl Velocity

Multiplying the equation for v^θ by $r\overline{\widehat{v^\theta}}$ yields

$$\int_0^1 \left|\frac{d}{dr}(r\widehat{v^\theta})\right|^2 \frac{1}{r}\,dr + \xi^2 \int_0^1 |\widehat{v^\theta}|^2 r\,dr = \Re \int_0^1 \widehat{F^\theta}\,\overline{\widehat{v^\theta}}\,r\,dr,$$

and

$$\frac{2\Phi}{\pi}\xi \int_0^1 (1-r^2)|\widehat{v^\theta}|^2 r\,dr = \Im \int_0^1 \widehat{F^\theta}\,\overline{\widehat{v^\theta}}\,r\,dr,$$

Multiplying the equation for v^θ by $r(\mathcal{L}-\xi^2)\overline{\widehat{v^\theta}}$ yields

$$\int_0^1 \left(|\mathcal{L}\widehat{v^\theta}|^2 + 2\xi^2 \left|\frac{1}{r}\frac{d}{dr}(r\widehat{v^\theta})\right|^2 + \xi^4|\widehat{v^\theta}|^2\right) r\,dr \le C\int_0^1 |\widehat{F^\theta}|^2 r\,dr.$$

Hence,

$$\|\mathbf{v}^\theta\|_{H^2(\Omega)} \le C\|\mathbf{F}^\theta\|_{L^2(\Omega)}. \tag{31}$$

Nonlinear Problem and Large Solutions

- Nonlinear stability can be obtained via a fixed point argument.
- In fact, we have the following results on the existence and uniqueness of large solutions.

  Theorem 3 Assume that $\mathbf{F} \in L^2(\Omega)$ and $\mathbf{F} = \mathbf{F}(r, z)$ is axisymmetric. There exist two independent constants $\Phi_0 \geq 1$ and C_0, such that if

$$\Phi \geq \Phi_0 \quad \text{and} \quad \|\mathbf{F}\|_{L^2(\Omega)} \leq \Phi^{\frac{1}{96}}, \tag{32}$$

  the problem for Navier-Stokes system has a unique axisymmetric solution $\mathbf{u}$, satisfying

$$\|\mathbf{u} - \bar{\mathbf{U}}\|_{H^{\frac{19}{12}}(\Omega)} \leq C\Phi^{\frac{1}{96}}$$

  and

$$\|u^r\|_{L^2(\Omega)} \leq C\Phi^{-\frac{15}{32}}. \tag{33}$$

Key Ingredients to Get Large Solutions-I

If ψ satisfies the equation

$$i\xi \bar{U}(r)(\mathcal{L}-\xi^2)\hat{\psi} - (\mathcal{L}-\xi^2)^2\hat{\psi} = \widehat{\partial_r F^z}. \tag{34}$$

Then one has

$$|\xi| \int_0^1 \frac{\bar{U}(r)}{r} \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 dr \le \int_0^1 \widehat{F^z} \frac{d}{dr}(r\overline{\hat{\psi}})\, dr.$$

Thus

$$\Phi\xi^2 \int_0^1 |\hat{\psi}|^2 r\, dr \le C\xi^2 \int_0^1 \int_0^1 \frac{\bar{U}(r)}{r} \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 dr \le C \int_0^1 |\widehat{F^z}|^2 r\, dr,$$

and consequently,

$$\|v^r\|_{L^2(\Omega)} = \left(\int_{\mathbb{R}} \xi^2 \int_0^1 |\hat{\psi}|^2 r dr d\xi \right)^{\frac{1}{2}} \le C\Phi^{-\frac{1}{2}} \|F^z\|_{L^2(\Omega)}.$$

Key Ingredients to Get Large Solutions-II

If ψ satisfies the equation

$$i\xi \bar{U}(r)(\mathcal{L}-\xi^2)\hat{\psi} - (\mathcal{L}-\xi^2)^2\hat{\psi} = \widehat{\partial_z F^r}. \tag{35}$$

Then one has

$$\begin{aligned}\Phi|\xi| \int_0^1 \frac{\bar{U}(r)}{r}\left|\frac{d}{dr}(r\hat{\psi})\right|^2 dr \le \left|\int_0^1 \xi \widehat{F^r}\bar{\hat{\psi}} r\, dr\right| \\ \le |\xi| \left(\int_0^1 |\widehat{F^r}|^2 r dr\right)^{1/2} \left(\xi^2 \int_0^1 \frac{\bar{U}(r)}{r}\left|\frac{d}{dr}(r\hat{\psi})\right|^2 dr\right)^{1/2}.\end{aligned}$$

Thus

$$\int_0^1 |\hat{\psi}|^2 r dr \le \Phi^{-2} \int_0^1 |\widehat{F^r}|^2 r dr.$$

Key Ingredients to Get Large Solutions-III

Furthermore, one has

$$\Phi|\xi|^2 \int_0^1 \frac{\bar{U}(r)}{r} \left| \frac{d}{dr}(r\hat{\psi}) \right|^2 dr \leq \left| \int_0^1 \xi^2 \widehat{F^r} \bar{\hat{\psi}} r \, dr \right| \leq C \int_0^1 |\widehat{F^r}|^2 r dr.$$

Again, we have

$$\|v^r\|_{L^2(\Omega)} = \left(\int_{\mathbb{R}} \xi^2 \int_0^1 |\hat{\psi}|^2 r \, dr d\xi \right)^{\frac{1}{2}} \leq C\Phi^{-\frac{1}{2}} \|F^r\|_{L^2(\Omega)}.$$

Therefore, we always have

$$\|v^r\|_{L^2(\Omega)} \leq C\Phi^{-\frac{1}{2}} \|\mathbf{F}\|_{L^2(\Omega)}.$$

Using the interpolation, we get the smallness of $\|v^r\|_{H^s(\Omega)}$ for $s < \frac{5}{3}$. Similarly, we can get the smallness for $\partial_z v^\theta$, $\partial_z \omega^\theta$.

Key Ingredients to Get Large Solutions-IV

The nonlinear equation for ψ has the form

$$\begin{aligned}&\bar{U}(r)\partial_z(\mathcal{L}+\partial_z^2)\psi-(\mathcal{L}+\partial_z^2)^2\psi\\ =&f-(v^r\partial_r+v^z\partial_z)\omega^\theta+\frac{2}{r}v^\theta\partial_z v^\theta+\frac{v^r}{r}\omega^\theta.\end{aligned}$$

The swirl velocity $\mathbf{v}^\theta=v^\theta\mathbf{e}_\theta$ satisfies

$$\bar{U}(r)\partial_z\mathbf{v}^\theta-\Delta\mathbf{v}^\theta=\mathbf{F}^\theta-(v^r\partial_r+v^z\partial_z)\mathbf{v}^\theta-\frac{v^r}{r}\mathbf{v}^\theta. \tag{36}$$

Observation: The nonlinear terms are small.

Asymptotic Behavior

Theorem 4 Assume that $\mathbf{F} = \mathbf{F}(r, z)$ is axisymmetric. There exists a constant α_0 depending only on Φ, such that if $\mathbf{F} = \mathbf{F}(r, z)$ satisfies

$$\|e^{\alpha z}\mathbf{F}\|_{L^2(\Omega)} < +\infty \tag{37}$$

with some $\alpha \in (-\alpha_0, \alpha_0)$, and $\mathbf{u}$ is an axisymmetric solution to the steady Navier-Stokes system in a pipe, which also satisfies

$$\|\mathbf{u} - \bar{\mathbf{U}}\|_{H^2(\Omega)} < +\infty, \tag{38}$$

then one has

$$\|e^{\alpha z}(\mathbf{u} - \bar{\mathbf{U}})\|_{H^2(\Omega)} < +\infty. \tag{39}$$

Remarks on Asymptotic Behavior

- Similar results were proved when Φ is small: Horgan-Wheeler(1978), Ames-Payne(1989), Galdi, $\cdots$
- The key point of Theorem 4 is that there is neither smallness assumption on the flux Φ nor the smallness on the deviation of $\mathbf{u}$ with $\bar{\mathbf{U}}$.
- It is clear that the solutions obtained in Theorem 3 must converge to Hagen-Poiseuille flows exponentially fast, if $\mathbf{F}$ decays exponentially fast.
- The convergence rate of v^{θ} can achieve $\Phi^{\frac{1}{2}}$.

Asymptotic Behavior for Small Perturbed Solutions

Proposition Assume that $\mathbf{F} \in L^2(\Omega)$, and $\mathbf{F} = \mathbf{F}(r, z)$ is axisymmetric. There exists a constant ϵ_0, independent of $\mathbf{F}$ and Φ, and a constant $\alpha_0 (\leq 1)$ depending on Φ, such that

$$\|e^{\alpha z}\mathbf{F}\|_{L^2(\Omega)} < +\infty, \quad \text{with some } \alpha \in (-\alpha_0, \alpha_0), \tag{40}$$

and $\mathbf{u}$ is an axisymmetric solution to the steady Navier-Stokes system, satisfying

$$\|\mathbf{u} - \bar{\mathbf{U}}\|_{H^2(\Omega)} \leq \epsilon_0, \tag{41}$$

then the solution satisfies

$$\|e^{\alpha z}(\mathbf{u} - \bar{\mathbf{U}})\|_{H^2(\Omega)} \leq C\|e^{\alpha z}\mathbf{F}\|_{L^2(\Omega)}. \tag{42}$$

Observation for Small Perturbed Solutions

Let $\mathbf{v} = \mathbf{u} - \bar{\mathbf{U}}$, and ψ is the corresponding stream function. Then

$$\begin{aligned}
&U(r)\partial_z(\mathcal{L}+\partial_z^2)(e^{\alpha z}\psi) - (\mathcal{L}+\partial_z^2)^2(e^{\alpha z}\psi)\\
=&e^{\alpha z}f - (v^r\partial_r + v^z\partial_z)(\mathcal{L}+\partial_z^2)(e^{\alpha z}\psi) + \frac{v^r}{r}(\mathcal{L}+\partial_z^2)(e^{\alpha z}\psi) - 2\omega^r\frac{e^{\alpha z}v^\theta}{r}\\
&+ U(r)\left[\alpha\mathcal{L}(e^{\alpha z}\psi) + \alpha^3 e^{\alpha z}\psi - 3\alpha^2\partial_z(e^{\alpha z}\psi) + 3\alpha\partial_z^2(e^{\alpha z}\psi)\right]\\
&- \left[4\alpha\partial_z^3(e^{\alpha z}\psi) - 6\alpha^2\partial_z^2(e^{\alpha z}\psi) + 4\alpha^3\partial_z(e^{\alpha z}\psi) - \alpha^4 e^{\alpha z}\psi\right]\\
&- \left[4\alpha\mathcal{L}\partial_z(e^{\alpha z}\psi) - 2\alpha^2\mathcal{L}(e^{\alpha z}\psi)\right] + \left(v^r\partial_r - \frac{v^r}{r}\right)\left[2\alpha\partial_z(e^{\alpha z}\psi) - \alpha^2 e^{\alpha z}\psi\right]\\
&+ \alpha v^z\mathcal{L}e^{\alpha z}\psi + v^z\left[3\alpha\partial_z^2(e^{\alpha z}\psi) - 3\alpha^2\partial_z(e^{\alpha z}\psi) - \alpha^3 e^{\alpha z}\psi\right],
\end{aligned}$$

and

$$\begin{aligned}
U(r)\partial_z(e^{\alpha z}\mathbf{v}^\theta) - \Delta(e^{\alpha z}\mathbf{v}^\theta) &= e^{\alpha z}\mathbf{F}^\theta - (\mathbf{v}^*\cdot\nabla)(e^{\alpha z}\mathbf{v}^\theta) - \frac{v^r}{r}(e^{\alpha z}\mathbf{v}^\theta)\\
&+ U(r)\alpha e^{\alpha z}\mathbf{v}^\theta - 2\alpha\partial_z(e^{\alpha z}\mathbf{v}^\theta) + \alpha^2 e^{\alpha z}\mathbf{v}^\theta + v^z\alpha e^{\alpha z}\mathbf{v}^\theta.
\end{aligned}$$

<u>Observation:</u> The right hand side is either nonlinear small or has a small factor α.

Observation for the General Solutions

Choose a smooth cut-off function $\eta(z)$ satisfying

$$\eta(z) = \begin{cases} 0, & z \leq L, \\ 1, & z \geq L+1. \end{cases}$$

Note that $(\psi, \mathbf{v}^\theta)$ satisfy

$$U(r)\partial_z(\mathcal{L}+\partial_z^2)(\eta\psi) - (\mathcal{L}+\partial_z^2)^2(\eta\psi) = \eta f + \tilde{f}$$
$$- \partial_r\left[v^r(\mathcal{L}+\partial_z^2)(\eta\psi)\right] - \partial_z\left[v^z(\mathcal{L}+\partial_z^2)(\eta\psi)\right] + \partial_z\left[\frac{v^\theta}{r}\eta v^\theta\right],$$

and

$$U(r)\partial_z(\eta\mathbf{v}^\theta) - \Delta(\eta\mathbf{v}^\theta) = \eta\mathbf{F}^\theta + \tilde{\mathbf{F}}^\theta - (v^r\partial_r + v^z\partial_z)(\eta\mathbf{v}^\theta) - \frac{v^r}{r}(\eta\mathbf{v}^\theta).$$

<u>Observation:</u> The terms in $\tilde{f}$ and $\tilde{\mathbf{F}}^\theta$ always contain the derivative of η so that they are finite.

Axially Symmetric Flows under Navier Boundary Conditions

Navier boundary conditions

$$\mathbf{v}\cdot\mathbf{n}=0 \quad \text{and} \quad 2\mathbf{n}^t D(\mathbf{v})\tau=\alpha\mathbf{v}\cdot\tau, \quad \text{when } r=1, \tag{43}$$

where

$$D(\mathbf{v})_{ij}=\frac{\partial_i u^j+\partial_j u^i}{2}.$$

The Poiseuille flows can be written as

$$\bar{U}(r)=\frac{\Phi}{\pi}\cdot\frac{2\alpha}{4-\alpha}r^2+\frac{\Phi}{\pi}\cdot\frac{4-2\alpha}{4-\alpha}.$$

- Using the similar idea, we can obtain the similar and better results for axially symmetric flows with Navier slip boundary conditions.
- Stability results: Ding, Li and Xin(2018), etc.

Summary and Further Problems

Summary

- Viscous flows in pipes

Problems:

- Global uniqueness/Liouville type theorem
- Viscous Steady flows in general nozzles
- Hydrodynamical stability and instability of Poiseuille flows
- ...

Thanks!