Global weak solutions to a three-dimensional compressible non-Newtonian fluid with vacuum

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outline

- Models
- Motivations
- Main results
- Key points of Proofs

Models

The equations of a compressible viscous barotropic fluid in $(x,t) \in \Omega \times \mathbb{R}^+$ have the following form

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}(\mathbb{P}) + \rho f. \end{cases} \quad (1)$$

ρ-density; $u = (u_1, u_2, u_3)$-velocity; $\mathbb{P}$-the stress tensor; f-the vector of external mass forces; the operators div and ∇ act with respect to the space variables x.

The initial data is given by

$$(\rho, \rho u)|_{t=0} = (\rho_0, m_0)(x), \ \ x \in \Omega,$$

and the no-slip boundary condition on the velocity

$$u \mid_{\partial\Omega} = 0.$$

Models

The system (1) must be closed by some constitutive equation for the stresses $\mathbb{P}$. Taking the Stokes axioms for the (only) criterion of its "physical validity," we restrict ourselves to constitutive relations of the following form

$$\mathbb{P} = \sum_{k=0}^{2} \alpha_k(\rho, \mathrm{div} u, |\mathbb{D}u|^2)\mathbb{D}^k u. \tag{2}$$

$\mathbb{D}u$-the deformation velocity tensor with components

$D_{ij}u = \frac{1}{2}(\partial_j u_i + \partial_i u_j);$

$|\mathbb{D}u|^2 \equiv \mathbb{D}u : \mathbb{D}u = \sum_{i,j=1}^{3} (D_{i,j}u)^2.$

One particular case of equation (3)

$$\mathbb{P} = -p(\rho) + \lambda(|\mathrm{div} u|^2)\mathrm{div} u\mathbb{I} + 2\mu(|\mathbb{D}u|^2)\mathbb{D}u \tag{3}$$

which is a natural generalization of the constitutive relation in the classical fluid model.

Motivations

The incompressible non-Newtonian fluids

$$\begin{cases} \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(\Gamma) + \nabla P = \rho f, \\ \text{div} u = 0 \end{cases}$$

where Γ denotes the viscous stress tensor and

$$\Gamma_{ij} = (\mu_0 + \mu_1 |\mathbb{D}u|)^{r-2} \mathbb{D}_{i,j} u$$

with $\mu_0 > 0, \mu_1 > 0$ are constants.

This form of Γ is proposed by O.A. Ladyzhenskaya 1970.

Motivations

The incompressible non-Newtonian fluids

—Existence of weak solutions

Ladyzhenskaya, Lions, Nečas, Zhikov, Kaniel, Frehse, Málek, Steinhauer, Boling Guo......

—The global attractor

Boling Guo, Guoguang Lin, Yadong Shang, Caidi Zhao, Yongsheng Li,......

The compressible non-Newtonian fluids

$$\begin{cases} \rho_t + \mathsf{div}(\rho u) = 0, \\ (\rho u)_t + \mathsf{div}(\rho u \otimes u) - \mathsf{div}(\Gamma) + \nabla P = \rho f. \end{cases}$$

where Γ denotes the viscous stress tensor and

$$\Gamma_{ij} = (\mu_0 + \mu_1 |\mathbb{D}u|)^{r-2} \mathbb{D}_{i,j} u$$

with $\mu_0 \geq 0, \mu_1 > 0$ are constants.

Motivations

These models are called :

- Newtonian, for $\mu_0 > 0, \mu_1 = 0$;
- Rabinowitsch, for $\mu_0, \mu_1 > 0, r = 4$;
- Eills, for $\mu_0, \mu_1 > 0, r > 2$;
- Ostwald-de Waele, for $\mu_0 = 0, \mu_1 > 0, r > 4$;
- Bingham, for $\mu_0, \mu_1 > 0, r = 1$;

For $\mu_0 = 0$, if $r < 2$, it is a pseudo-plastic fluid; if $r > 2$, it is a dilant fluid;

In the view of physics:

$1 < r < 2$: shear thinning fluid
$r > 2$: shear thickening fluid.

Motivations

◇ **One-dimension**

—Local existence of strong/classical solution

- Hongjun Yuan and his team, Qin Yumin,
  Guo Zhenhua, Fang Li, Wang Yuxin......

—Asymptotic stability/Large-time behavior of solution

- Shi-Wang-Zhang (2014): Asymptotic stability
- Guo-Fang (2016): Zero dissipation limit to rarefaction wave with vacuum
- Guo-Dong-Liu (2019): Large-time behavior of solution to an inflow problem on the half space
- Guo-Su-Liu: The existence and limit behavior of the shock layer for 1D steady compressible non-Newtonian fluids
- other results ... ...

Motivations

◇ **Multi-dimension**

—Existence of weak solution

Feireisl, Liao and Málek, Zhikov and Pastukhova, Mamontov, ...

Motivations

Feireisl-Liao-Málek considered the following compressible non-Newtonian fluid in $(x,t) \in \Omega \times \mathbb{R}_+$,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) \\ \qquad = \mathrm{div}(2\mu_0(1 + |\mathbb{D}^d u|^2)^{\frac{r-2}{2}} \mathbb{D}^d u + \eta(\mathrm{div} u)\mathrm{div} u \mathbb{I}) \end{cases} \tag{4}$$

where i) $\mathbb{D}^d u = \mathbb{D}u - \frac{1}{3}(tr\mathbb{D}u)\mathbb{I}$, $\mu_0 > 0$ is a constant, $r \in [\frac{11}{5}, +\infty)$;

ii) the bulk viscosity coefficient η is a continuous function of div$u$, $\eta(z) : (-\frac{1}{b}, \frac{1}{b}) \to [0, +\infty)$ such that there is a convex potential $\Lambda : \mathbb{R} \to [0, \infty]$

$$\begin{cases} \Lambda(0) = 0, & \Lambda'(z) = z\eta(z), \\ \Lambda(z) \to \infty & \text{if } z \to \pm\frac{1}{b}, \\ \Lambda(z) = \infty & \text{if } |z| \geqslant \frac{1}{b}; \end{cases}$$

iii) the pressure $p = p(\rho)$ and the Helmholtz free energy $\psi = \psi(\rho)$ satisfy

$p = \rho^2\psi'(\rho)$, $p \in C[0,\infty) \cap C^1(0,+\infty)$, $p(0) = 0$, $p'(\rho) > 0$ for $\rho > 0$.

Motivations

The definition of weak solution in the work Feireisl-Liao-Málek

A pair of functions (ρ, u) is said to be a weak solution to the problem (4) on $(0,T)$ for any fixed $T>0$ if the following conditions hold:

- $\rho \in C([0,T];L^1(\Omega)) \cap L^\infty(\Omega\times(0,T))$,
  $u \in L^r(0,T;W_0^{1,r}(\Omega))$, $\eta(|\mathrm{div} u|)|\mathrm{div} u|^2 \in L^1(\Omega\times(0,T))$;
- The equation of continuity in (4) is satisfied in $\mathcal{D}'(\Omega\times(0,T))$;
- The following weak formulation of the momentum equation

$$[\tfrac{1}{2}\int_\Omega \rho|u|^2dx]\Big|_0^\tau - [\int_\Omega \rho u\cdot\varphi dx]\Big|_0^\tau + \int_0^\tau\int_\Omega[\rho u\cdot\partial_t\varphi + \rho u\otimes u : \nabla\varphi]dxdt$$

$$+\int_0^\tau\int_\Omega |\mathbb{D}u|^{r-1}\mathbb{D}u : \mathbb{D}(u-\varphi)dxdt + \int_0^\tau\int_\Omega p(\rho)\mathrm{div}(\varphi-u)dxdt$$

$$\leqslant \underline{\int_0^\tau\int_\Omega[\Lambda(\mathrm{div}\varphi)-\Lambda(\mathrm{div}u)]dxdt} \text{ (control the term } \eta(|\mathrm{div}u|)\mathrm{div}u)$$

$$\eta(\mathrm{div}u) : (-\tfrac{1}{b}, -\tfrac{1}{b}) \to [0,+\infty)$$

for any $\tau\in[0,T]$ and any test function $\varphi \in C_c^\infty(\Omega\times[0,T])$.

Motivations

Feireisl-Liao-Málek (2015) showed the large-data existence result of weak solutions to the initial-boundary problem to the system (4) with nonlinear constitutive equations that guarantee that the divergence of the velocity field remains bounded, provided the initial density is without vacuum.

Motivations

Zhikov-Pastukhova (2009) considered a compressible fluid in $(x,t) \in \Omega \times \mathbb{R}_+$, described by

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma \\ \qquad = \mathrm{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u + \nu|\mathrm{div} u|^{r-2}\mathbb{I}) \end{cases} \tag{5}$$

where $\nu \geqslant 0$ is a constant, $\gamma > 1$, $r > 1$.

The definition of weak solution in the work Zhikov and Pastukhova

A pair of functions (ρ, u) is said to be a weak solution to the problem (5) on $(0, T)$ for any fixed $T > 0$ if the following conditions hold:

(1) $\rho \ \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega))$,

$u \in L^r(0, T; W_0^{1,r}(\Omega)), \ \rho u \in L^\infty(0, T; L^1(\Omega))$;

(2) The equations (5) is satisfied in $\mathcal{D}'(\Omega \times (0, T))$;

(3) $\lim\limits_{t \to 0} \rho = \rho_0$ in $L^1(\Omega)$,

$\lim\limits_{t \to 0} \int_\Omega \rho u \cdot \varphi dx = \int_\Omega m_0 \cdot \varphi dx \ \ \forall \varphi \in C_0^\infty(\Omega)$.

Motivations

Zhikov and Pastukhova (2009) proved that the initial-boundary problem to the system (6) admits a weak solution such that

$$\rho u^2 \in L^{\frac{r}{r-1}}(\Omega \times (0,T)),\ \ \rho^\gamma \in L^{\frac{r}{r-1}}(\Omega \times (0,T))$$

provided that $\rho_0 \in L^\gamma(\Omega),\ \frac{m_0}{\rho_0} \in L^1(\Omega),\ \gamma > \frac{3}{2},\ r \geqslant 3.$

Motivations

Mamontov (1999) considered a compressible fluid in $(x,t) \in \Omega \times \mathbb{R}_+$, described by

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) \\ \qquad = \mathrm{div}(2\mu(|\mathbb{D}u|^2)\mathbb{D}u + \lambda(|\mathrm{div} u|^2)\mathrm{div} u \mathbb{I}) \end{cases} \tag{6}$$

where $\mu(s) = \exp(s^{\epsilon_0})$ ($\epsilon_0 > 0$ a constant), $\lambda(s) = \exp(\sqrt{s})$.

Mamontov showed the existence of global solutions of multidimensional equations of motion of a compressible non-Newtonian fluid in Bürgers approximation (in the absence of pressure), on the basis of the techniques of Orlicz spaces.

Global weak solution to the multi-dimensional compressible Navier-Stokes equations for general initial data with finite energy:

—Lions (Oxford 1998), Feireisl-Novotny-Petzeltová (JMFM, 2001), Jiang-Zhang (CMP, 2001), ...

Question: For general initial data with finite energy, what about the existence of global weak solution to the multi-dimensional compressible non-Newtonian fluid containing vacuum for the general case $r > 1$?

Main results

Consider the initial-boundary value problem for the isentropic compressible non-Newtonian fluid with vacuum

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p \\ \quad = \mathrm{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u + \eta(|\mathrm{div} u|)\mathrm{div} u \mathbb{I}), \end{cases} \tag{7}$$

with the initial data

$$(\rho, \rho u)|_{t=0} = (\rho_0, m_0)(x), \ \ x \in \Omega \tag{8}$$

and the no-slip boundary condition on the velocity

$$u \ |_{\partial\Omega} = 0. \tag{9}$$

Notice: This model can describe the motion of electrons in an electric field.

Definition of weak solution

Definition A pair of functions (ρ, u) is said to be a finite energy weak solutions to the problem (7)-(9) on $(0, T)$ for any fixed $T > 0$ if the following conditions hold:

- $\rho \geqslant 0, \ \rho \ \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega))$, $u \in L^r(0,T; W_0^{1,r}(\Omega)), \ \eta(|\text{div}u|)|\text{div}u|^2 \in L^1(\Omega \times (0,T))$;
- The equations (7) hold in $\mathcal{D}'(\Omega \times (0,T))$ and

$$\int_0^\tau \int_\Omega [\rho \partial_t \varphi + \rho u \cdot \nabla \varphi] dx dt = [\int_\Omega \rho \varphi dx]|_0^\tau$$

  for any $\tau \in [0,T]$ and any test function $\varphi \in C^\infty(\Omega \times [0,T])$ with $\varphi(x,0) = \varphi(x,T) = 0$ for $x \in \Omega$;
- The functions ρ and ρu satisfy the initial conditions in the weak sense.

Remark

- Any weak solution in Definition satisfy the equations (7) hold in $\mathcal{D}'(\Omega \times (0,T))$, which is different from the definition of weak solution in the work Feireisl-Liao-Málek.
- Any weak solution in Definition satisfies the following weak formulation of the momentum equation

$$[\frac{1}{2}\int_\Omega \rho|u|^2 dx]\Big|_0^\tau - [\int_\Omega \rho u\cdot\varphi dx]\Big|_0^\tau + \int_0^\tau \int_\Omega [\rho u\cdot\partial_t\varphi + \rho u\otimes u : \nabla\varphi] dxdt$$

$$+ \int_0^\tau \int_\Omega |\mathbb{D}u|^{r-1}\mathbb{D}u : \mathbb{D}(u-\varphi) dxdt + \int_0^\tau \int_\Omega p(\rho)\mathrm{div}(\varphi - u) dxdt$$

$$\leqslant \int_0^\tau \int_\Omega [\Lambda(\mathrm{div}\varphi) - \Lambda(\mathrm{div}u)] dxdt$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty(\Omega \times [0,T])$, where $\Lambda'(z) = \eta(z)z$ and $\Lambda''(z) \geqslant 0$.

Theorem 1 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:

(i) the pressure $p(\rho)$ is given by $p(\rho) = \rho^\gamma$ with the adiabatic exponent $\gamma > \frac{3}{2}$;

(ii) the initial data satisfy

$$\begin{cases} \rho_0 \in L^\gamma(\Omega), \rho_0 \geqslant 0 \text{ on } \Omega, \\ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega); \end{cases}$$

(iii) the positive constants r and q satisfy the case that $\frac{12}{5} \leqslant r < 3$ and $q > \max\{\gamma, 9\}$, or the case that $r \geqslant 3$ and $q > 1$.

Then, the initial-boundary value problem (7)-(9) admits a finite energy weak solution (ρ, u) on $\Omega \times (0, T)$ for any given $T > 0$.

Theorem 2 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:

(i) the pressure $p(\rho)$ is given by

$$\begin{cases} p'(s) \geqslant a_1 s^{\gamma-1} \text{ for all } s > 0, & p(s) \leqslant a_2 s^{\gamma} \text{ for all } s \geqslant 0, \\ p \in C[0,\infty) \cap C^1(0,\infty), & p(0) = 0 \end{cases} \tag{10}$$

with the adiabatic exponent $\gamma > \frac{3}{2}$;

(ii) the initial data satisfy

$$\begin{cases} \rho_0 \in L^{\gamma}(\Omega), \rho_0 \geqslant \underline{\rho} > 0 \text{ on } \Omega, \\ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega); \end{cases}$$

(iii) the positive constants p and q satisfy the case that $\frac{11}{5} \leqslant r < 3$ and $q > \max\{\gamma, 9\}$, or the case that $r \geqslant 3$ and $q > 1$.

Then, the initial-boundary value problem (7)-(9) admits a finite energy weak solution (ρ, u) on $\Omega \times (0,T)$ for any given $T > 0$.

Remark

(i) The solution constructed in Theorem 1 -2 admits that

$$\rho^\gamma \in L^{\frac{q+1}{q}}(\Omega\times(0,T)) \text{ (when } \frac{12}{5}\leqslant r<3 \text{ and } q>\max\{\gamma,9\}),$$
$$\text{or } \rho^\gamma \in L^{\frac{r+1}{r}}(\Omega\times(0,T)) \text{ (when } r\geqslant 3 \text{ and } q>1),$$
$$\mathrm{div} u\in L^{q+1}(\Omega\times(0,T)) \text{ and } \nabla u\in L^r(\Omega\times(0,T)).$$

(ii) The solution constructed in Theorem 1 and Theorem 2 will satisfy the continuity equation in the sense of re-normalized solutions.

(iii) Our results also hold for the bulk viscosity coefficient $\eta(|\mathrm{div} u|)\sim|\mathrm{div} u|^{q-1}$, where the symbol $\sim$ refers that there exist positive constants C_1 and C_2 such that

$$C_1|\mathrm{div} u|^{q-1}\leqslant\eta(|\mathrm{div} u|)\leqslant C_2|\mathrm{div} u|^{q-1}$$

and $\eta(z)+z\eta'(z)>0$ holds for any $z>0$.

Main difficulties and the countermeasure in proof of Theorems

(1) The initial density containing vacuum and strong degeneracy of the term $\text{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u)$ in momentum equations

Inspired by Jiang-Zhang-2001 and Chapter 7 in Feireisl-2004, we introduce an approximate problem

$$\begin{cases} \partial_t\rho + \text{div}(\rho u) = \epsilon\Delta\rho, \\ \partial_t(\rho u) + \text{div}(\rho u\otimes u) + \nabla(p+\delta\rho^\beta) + \epsilon\nabla u\cdot\nabla\rho \\ \qquad\qquad = \text{div}((\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u + \eta(|\text{div}u|)\text{div}u\mathbb{I}) \end{cases} \tag{11}$$

with the initial-boundary conditions

$$\begin{cases} \nabla\rho\cdot n|_{\partial\Omega} = 0, & \rho|_{t=0} = \rho_{0,\delta}, \\ u|_{\partial\Omega} = 0, & \rho u|_{t=0} = m_{0,\delta}, \end{cases} \tag{12}$$

(2) The strong nonlinearity for the pressure term $p(\rho)$ and the term $\text{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u)$.

The difficulty comes from the nonlinear term

$$(\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u + \eta(|\text{div}u|)\text{div}u\mathbb{I}$$

in the approximate problem.

Recall that $\eta(z) = |z|^{q-1}$, $\Lambda'(z) = \eta(z)z$ and $\Lambda''(z) \geqslant 0$, one can deduce that

$$\overline{\eta(\text{div}u)\text{div}u} = \eta(\text{div}u)\text{div}u.$$

So we need focus on the first term of the above nonlinear term.

Sketch of Theorem 1's proof

Step 1. Limit in the Galerkin approximation

The approximate problem (11)-(12) with fixed positive parameters ϵ and δ can be solved by means of a modified Faedo-Galerkin method.

Step 2. The vanishing limits of the artificial viscosity $\epsilon \to 0$.

The common difficulty in the Step 1 and the Step 2 :

The nonlinear term

$$(\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u$$

in the approximate problem.

The key tool to deal with the nonlinear term above

Let $A_\epsilon(x,\xi)$ and $A(x,\xi)$ be Carathéodory vector functions. $A_\epsilon(x,\xi),\ A(x,\xi):\ \Omega\times\mathbb{R}^d\to\mathbb{R}^d$, where Ω is a bounded domains in $\mathbb{R}^d$. The Carathéodory property means continuity with respect to $\xi\in\mathbb{R}^d$ for a.e. $x\in\Omega$ and measurability with respect to x for any ξ. These vector functions are assumed to satisfy the minimal monotonicity and convergence conditions

$$\begin{aligned}(A_\epsilon(x,\xi)-A_\epsilon(x,\eta))\cdot(\xi-\eta)\geqslant 0, &\quad A_\epsilon(x,0)\equiv 0\\ |A_\epsilon(x,\xi)|\leqslant c_0(|\xi|)<\infty, &\quad \lim_{\epsilon\to 0}A_\epsilon(x,\xi)=(x,\xi)\end{aligned}$$

for a.e. $x\in\Omega$ and any $\xi\in\mathbb{R}^d$.

Lemma

Suppose that $v_\epsilon \rightharpoonup v, \quad A_\epsilon(x, v_\epsilon) \rightharpoonup z$ *in* $L^1(\Omega)$. *Let* $K \subset \Omega$ *be a measurable set such that* $z \cdot v \in L^1(K)$. *Then*

$$\liminf_{\epsilon \to 0} \int_K A_\epsilon(x, v_\epsilon) \cdot v_\epsilon dx \geqslant \int_K z \cdot v dx$$

and, in the case of equality,

$$z|_K = A|_K, \quad A = A(x, v).$$

We employ the following weak formulation of the momentum equation in the approximate problem

$$
\begin{aligned}
&[\frac{1}{2}\int_{\Omega}\rho|u|^2dx]\Big|_0^{\tau} - [\int_{\Omega}\rho u\cdot\varphi dx]\Big|_0^{\tau} \\
&+\int_0^{\tau}\int_{\Omega}\left(\rho u\cdot\partial_t\varphi + \rho u\otimes u : \nabla\varphi\right)dxdt \\
&+\int_0^{\tau}\int_{\Omega}(p(\rho)+\delta\rho^{\beta})(\mathsf{div}\varphi - \mathsf{div}u)dxdt - \int_0^{\tau}\int_{\Omega}\epsilon\nabla\rho\cdot\nabla u\cdot\varphi dxdt \\
&+\int_0^{\tau}\int_{\Omega}\Big(\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2} - \overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u} : \mathbb{D}\varphi\Big)dxdt \\
&\leqslant \int_0^{\tau}\int_{\Omega}[\Lambda(\mathsf{div}\varphi)-\Lambda(\mathsf{div}u)]dxdt \text{ for } a.e.\tau\in[0,T] \qquad (13)
\end{aligned}
$$

The family of regularized Kernels

$$\eta_h(t) := \frac{1}{h}\mathbb{I}_{[-h,0]}(t) \text{ and } \eta_{-h}(t) := \frac{1}{h}\mathbb{I}_{[0,h]}(t)(h>0),$$

together with the cut-off functions

$$\xi_\sigma \in C_c^\infty(0,\tau), \quad 0 \leqslant \xi \leqslant 1, \quad \xi_\sigma(t) = 1$$

whenever $t \in [\sigma, \tau - \sigma], \quad \sigma > 0$. Noticing that $\eta_h * u = \frac{1}{h}\int_t^{t+h} u ds \in W^{1,r}(0,T; W_0^{1,r}(\Omega))$, we can take the quantities

$$\varphi_{h,\sigma} = \xi_\sigma \eta_{-h} * \eta_h * (\xi_\sigma u) \ (\sigma, h > 0)$$

as test functions in (13).

By careful calculation, we can arrive at

$$\int_0^\tau \int_\Omega \left(\overline{(\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} |\mathbb{D}u|^2} - \overline{(\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u} : \mathbb{D}u \right) dxdt \leqslant 0,$$

based on the fact that

$$\lim_{\sigma \to 0} \lim_{h \to 0} \int_0^\tau \int_\Omega (\Lambda(\mathsf{div}\varphi_{h,\sigma}) - \Lambda(\mathsf{div}u))\, dxdt = 0,$$

$$[\int_\Omega \rho u \cdot \varphi_{h,\sigma} dx]|_0^\tau = 0 \text{ (for all } \sigma, h > 0),$$

So together with Fatou's lemma,

$$\lim_{\sigma\to 0}\lim_{h\to 0}\int_0^\tau\int_\Omega\left(\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}-\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}:\mathbb{D}\varphi_{h,\sigma}\right)dxdt \geqslant 0.$$

it ensures that

$$\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u}=(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}\mathbb{D}u$$

and

$$\overline{(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2}=(\delta+|\mathbb{D}u|^2)^{\frac{r-2}{2}}|\mathbb{D}u|^2.$$

Step 3. The artificial pressure coefficient $\delta \to 0$.

3.1 The density estimates.

Lemma

There exists a positive constant C, *independence of* δ, *such that*

$$\int_0^T \int_\Omega (\rho_\delta^{\frac{q+1}{q}\gamma} + \delta \rho_\delta^{\beta + \frac{\gamma}{q}}) dxdt \leqslant C,$$

holds for the case that $\frac{11}{5} \leqslant r < 3$ *and* $q > \max\{\gamma, 9\}$, *and*

$$\int_0^T \int_\Omega (\rho_\delta^{\frac{r}{r-1}\gamma} + \delta \rho_\delta^{\beta + \frac{\gamma}{r-1}}) dxdt \leqslant C,$$

holds for the case that $r \geqslant 3$ *and* $q > 1$.

3.2 The amplitude of oscillations.

For the cut-off operators introduced in Feireisl-2004 and Jiang-Zhang-2001, we consider a family of functions

$$T_k(z) = kT(\frac{z}{k}) \text{ for } z \in \mathbb{R},\ k = 1, 2, \cdots \tag{14}$$

where $T \in C^\infty(\mathbb{R})$ is chosen so that

$$T(z) = z \text{ for } z \leqslant 1,\ T(z) = 2 \text{ for } z \geqslant 3,\ T \text{ concave.}$$

Lemma

There exists a positive constant C, independence of k, such that

$$\limsup_{\delta\to 0} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\frac{q+1}{q}\gamma}(\Omega\times(0,T))} \leqslant C$$

holds for the case that $\frac{12}{5} \leqslant r < 3$ *and* $q > \max\{\gamma, 9\}$,

$$\limsup_{\delta\to 0} \|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}(\Omega\times(0,T))} \leqslant C$$

holds for the case that $r \geqslant 3$ *and* $q > 1$.

Remark. The limit functions ρ and u satisfy the continuity equation $(4)_1$ in the sense of renormalized solutions.

3.3 The momentum equation.

The first key point is to prove $\overline{\rho^\gamma} = \rho^\gamma$. The following integrability properties of the limit functions ρ and u, play an important role, which are stated as

$$\rho^\gamma \in L^{\frac{q+1}{q}}(\Omega \times (0,T)) \ (\frac{12}{5} \leqslant r < 3 \text{ and } q > \max\{\gamma, 9\}),$$

$$\text{or} \quad \rho^\gamma \in L^{\frac{r}{r-1}}(\Omega \times (0,T)) \ (r \geqslant 3 \text{ and } q > 1),$$

$$\text{div} u \in L^{q+1}(\Omega \times (0,T)) \text{ and } \nabla u \in L^r(\Omega \times (0,T)),$$

$$\rho u^2 \in L^{\frac{r}{r-1}}(\Omega \times (0,T)).$$

Difficulty from vacuum in initial density in Theorem 1

The second key equality $\overline{|\mathbb{D}u|^r} = |\mathbb{D}u|^r$ is obtained by applying the technique in the following inequality

$$
\begin{aligned}
&[\frac{1}{2}\int_\Omega \rho|u|^2 dx]\Big|_0^\tau - [\int_\Omega \rho u\cdot\varphi dx]\Big|_0^\tau \\
&+\int_0^\tau\int_\Omega(\rho u\cdot\partial_t\varphi + \rho u\otimes u : \nabla\varphi)dxdt \\
&+\int_0^\tau\int_\Omega \rho^\gamma(\mathrm{div}\varphi - \mathrm{div}u)dxdt + \int_0^\tau\int_\Omega\left(\overline{|\mathbb{D}u|^r} - \overline{|\mathbb{D}u|^{r-2}\mathbb{D}u} : \mathbb{D}\varphi\right)dxdt \\
&\leqslant \int_0^\tau\int_\Omega\left(\Lambda(\mathrm{div}\varphi) - \Lambda(\mathrm{div}u)\right)dxdt \text{ for } a.e.\tau\in[0,T] \qquad (15)
\end{aligned}
$$

In Theorem 2, the pressure $p(\rho)$ is given by

$$\begin{cases} p'(s) \geqslant a_1 s^{\gamma-1} \text{ for all } s > 0, & p(s) \leqslant a_2 s^{\gamma} \text{ for all } s \geqslant 0, \\ p \in C[0,\infty) \cap C^1(0,\infty), & p(0) = 0 \end{cases}$$

with the adiabatic exponent $\gamma > \frac{3}{2}$; and the initial data satisfy

$$\begin{cases} \rho_0 \in L^{\gamma}(\Omega), \rho_0 \geqslant \underline{\rho} > 0 \text{ on } \Omega, \\ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega). \end{cases}$$

The approximate problem is still adopted as

$$\begin{cases} \partial_t \rho + \mathsf{div}(\rho u) = \epsilon \Delta \rho, \\ \partial_t(\rho u) + \mathsf{div}(\rho u \otimes u) + \nabla(p + \delta \rho^\beta) + \epsilon \nabla u \cdot \nabla \rho \\ \qquad\qquad = \mathsf{div}((\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u + \eta(|\mathsf{div} u|)\mathsf{div} u \mathbb{I}) \end{cases}$$

with the initial-boundary conditions

$$\begin{cases} \nabla \rho \cdot n|_{\partial\Omega} = 0, & \rho|_{t=0} = \rho_{0,\delta}, \\ u|_{\partial\Omega} = 0, & \rho u|_{t=0} = m_{0,\delta}, \end{cases}$$

The difference between proof of Theorem 2 and Theorem 1 is the pressure term.
The limit in the Galerkin approximation and the vanishing limits of the artificial viscosity $\epsilon \to 0$ can be settled by the similar way in proof of Theorem 1.

In the artificial pressure coefficient $\delta \to 0$, one can arrive at

$$\int_0^\tau \int_\Omega (\overline{p(\rho)}\text{div}u - \overline{p(\rho)\text{div}u})dxdt \leqslant 0 \text{ for a.a } \tau \in [0,T].$$

To deal with $\overline{\rho P(\rho)} - \rho P(\rho)$, the convexity of the function $zP(z)$ with $P(z) = \int_1^z \frac{p(s)}{s^2}ds$ and the initial density without vacuum play an important role.

Since the initial density is without vacuum, the convexity of the function $zP(z)$ implies that there exists a certain $\alpha > 0$ such that

$$\int_\Omega [\overline{\rho P(\rho)} - \rho P(\rho)]dx \geqslant \alpha \limsup_{\delta \to 0} \int_\Omega |\rho_\delta - \rho|^2 dx.$$

The constant $\alpha > 0$ in above inequality depends on the positive lower bound of the initial density This is different way to deal with the initial density being without vacuum.

Thank you for your attention!