Completeness for $\Sigma^0_{n+2}$ Sets

Adam Day (joint work with Andrew Marks)

May 29, 2019
Outline

1. Completeness Background
2. A Completeness Criterion
3. Applications to Decomposability
Let $\mathcal{X}$ be an uncountable Polish space and $E \subseteq \mathcal{X}$.

The Borel classes are closed under continuous preimages.

How can you show that $E$ is $\Sigma^0_n$-complete?

Give a $\Sigma^0_n$ definition of $E$ and show a complete $\Sigma^0_n$ set is continuously reducible to it.
Work in $2^\mathbb{N}$. Define:

- $\Gamma_1 = \{ x : \exists n(x(n) = 1) \}$.

- $\Gamma_{i+1}$ to be $\{ x \in 2^\mathbb{N} : \exists n(x^{[n]} \not\in \Gamma_i) \}$.

(where $x^{[n]}(i) = x(\langle n, i \rangle)$).

- It easy to show that $\Gamma_i$ is $\Sigma^0_i$ complete.

- For example, $\Gamma_3$ is

$$\{ x : \exists i \forall j \exists k(x(\langle i, \langle j, k \rangle \rangle) = 1) \}$$
Showing $\Gamma_i$ is Complete

Completeness for $\Gamma_3$

Let $E \subseteq 2^\mathbb{N}$ be $\Sigma^0_3$. Hence

$$E = \bigcup_{i} \bigcap_{j} \bigcup_{k} D_{\langle i,j,k \rangle}$$

where each $D_{\langle i,j,k \rangle}$ is basic clopen.

Define $h : \mathcal{X} \rightarrow 2^\mathbb{N}$ by

$$h(x)(\langle i,j,k \rangle) = \begin{cases} 1 & x \in D_{\langle i,j,k \rangle} \\ 0 & \text{otherwise.} \end{cases}$$

Then $h$ continuously reduces $E$ to $\Gamma_3$, so $\Gamma_3$ is $\Sigma^0_3$-complete.

But how about a set that isn’t as nice as $\Gamma_i$?
Idea: Fix $E \subseteq \mathcal{X}$. Construct $f : 2^\mathbb{N} \rightarrow \mathcal{X}$.

Use forcing with a countable collection of closed sets, and Cohen forcing to construct the reduction $f : 2^\mathbb{N} \rightarrow \mathcal{X}$.

- Use Cohen forcing to obtain $f(x) \in E$.
- Use the closed set forcing to obtain $f(x) \notin E$. 
Orthogonal Forcings

- Let $A_0$ be all basic open subsets of $\mathcal{X}$.
- Let $A_1$ be a countable collection of closed subsets of $\mathcal{X}$ that is
  - closed under finite intersection.
  - contains the closures of all basic open sets.
- Order $A_1$ under inclusion
- A generic filter in this partial order uniquely determines an element of $\mathcal{X}$.
- Write $\Vdash_{A_1} B$ if every $A_1$-generic element of $\mathcal{X}$ is a member of $B \subseteq \mathcal{X}$.
- The set of these generic elements of $\mathcal{X}$ is $\Pi^0_3$. 
Cohen forcing inside elements of $\mathcal{A}_1$.

- Let $C \in \mathcal{A}_1$.
- Define $\mathcal{A}_0 \upharpoonright C = \{ B \in \mathcal{A}_0 : B \cap C \neq \emptyset \}$.
- Define an ordering on $\mathcal{A}_0 \upharpoonright C$ by $B_0 \leq B_1$ if $B_0 \cap C \supseteq B_1 \cap C$.
- A generic filter in $\mathcal{A}_0 \upharpoonright C$ uniquely determines an element of $C$.
- Write $\Vdash_{\mathcal{A}_0 \upharpoonright C} B$ if every such element of $C$ is a member of $B \subseteq \mathcal{X}$.
- The set of generics for $\mathcal{A}_0 \upharpoonright C$ for all $C \in \mathcal{A}_1$ is $\Sigma_3^0$. 
Completeness for $\Sigma_3^0$ Sets

**Theorem (Abstraction of a folklore technique)**

Let $E$ be a subset of $\mathcal{X}$ a Polish space. Let $\mathcal{A}_1$ be a collection of closed subsets of $\mathcal{X}$ that is: closed under finite intersections, and contains the closures of all basic open sets. If

- $\models_{\mathcal{A}_1} \mathcal{X} \setminus E$.
- For all $C \in \mathcal{A}_1$, $\models_{\mathcal{A}_0 \upharpoonright C} E$

Then $E$ is $\Sigma_3^0$ complete.

**Proof Idea**

Define a reduction $h : 2^\omega \to \mathcal{X}$ such that $f^{-1}(E) = \{x \in 2^\omega : \exists n(x[n] \text{ is infinite})\}$.
Completeness for \( \Sigma^0_3 \) Sets

Fix \( x \) and define \( p^x_1, p^x_2, \ldots \).

- Assume \( p^x_n = ((C_0, C_1, \ldots, C_k), B) \) where each \( C_i \in A_1 \) with \( C_0 \supseteq C_1 \supseteq \ldots \supseteq C_k \), \( B \in A_0 \), radius of \( B \) is less than \( 2^{-n} \), and \( B \cap C_k \neq \emptyset \).

- If \( x(n) = 0 \) then define \( p^x_{n+1} \) map \( x \) to be

\[
((C_0, C_1, \ldots, C_k, C_{k+1}), \hat{B})
\]

where \( C_{k+1} \) meets the next dense open set inside \( A_1 \) and \( \hat{B} \subseteq B \).

- If \( x(n) = 1 \) and \( n = \langle e, m \rangle \) for some \( m \) with \( e \leq k \), then define \( p^x_{n+1} \) to be

\[
((C_0, C_1, \ldots, C_e), \hat{B})
\]

where \( \hat{B} \subseteq B_n \) that meets the next dense set inside \( A_0 \restriction C_e \).

- Define \( h(x) \) to be unique element of \( \bigcap_n \pi_2(p^x_n) \).
A Completeness Criterion for $\Sigma^0_n$ Sets

Theorem

Let $\mathcal{X}$ be a Polish space. Let $n \geq 3$. A set $E \subseteq \mathcal{X}$ is $\Sigma^0_n$ hard if and only if there is “densely closed” “suitable sequence” $A_0, \ldots, A_{n-2}$ such that:

1. $\models A_{n-2} \mathcal{X} \setminus E$.
2. $\models A_{n-3} \upharpoonright B \ E$ for all $B \in A_{n-2}$.

Suitable sequence means:

- $A_0$ is all basic open sets.
- $A_i$ is a countable collection of $\Pi^0_i$ sets.
- $A_i$ is closed under finite intersection (with itself and appropriate lower $A_j$’s).
- $A_{i+2}$ is expressible in terms of $A_i$. 
A Completeness Criterion for $\Sigma^0_n$ Sets

**Theorem**

Let $\mathcal{X}$ be a Polish space. Let $n \geq 3$. A set $E \subseteq \mathcal{X}$ is $\Sigma^0_n$ hard if and only if there is “densely closed” “suitable sequence” $A_0, \ldots, A_{n-2}$ such that:

1. $\models_{A_{n-2}} \mathcal{X} \setminus E$.
2. $\models_{A_{n-3}|_B} E$ for all $B \in A_{n-2}$.

1. means any generic element obtained by forcing with $A_{n-2}$ is in $\mathcal{X} \setminus E$.

2. means for any $B \in A_{n-2}$, any generic element obtained by forcing with $\{ C \in A_{n-3} : C \cap B \neq \emptyset \}$ is in $E$. 
Proof Idea

Easy direction:

- Show that $\Gamma_n$ has these properties.
- If $E$ is $\Sigma^0_n$ hard find $f : 2^\mathbb{N} \to \mathcal{X}$ such that $f^{-1}(E) = \Gamma_n$.
- Use $f$ to push forward these properties to $\mathcal{X}'$.

Hard direction:

- Run the $\Sigma^0_3$ construction on top of a “true-stages” argument.
- The case $n = 5$ is representative.

Assume we have a densely closed, suitable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that:

- $\models_{\mathcal{A}_3} \mathcal{X} \setminus E$.
- $\models_{\mathcal{A}_2 \upharpoonright B} E$ for all $B \in \mathcal{A}_3$. 
To show $E$ is $\Sigma^0_5$-hard will build a continuous function $h : 2^\mathbb{N} \rightarrow \mathcal{X}$ such that $h^{-1}(E) = \Gamma_5$.

- $\mathcal{P} = \{(A, B, C, D) \in A_0 \times A_1 \times A_2 \times A_3 : A \cap B \cap C \cap D \neq \emptyset\}$.

- Order $\mathcal{P}$ by $(A_0, B_0, C_0, D_0) \leq (A_1, B_1, C_1, D_1)$ if $A_0 \subseteq A_1, B_0 \subseteq B_1, C_0 \subseteq C_1, \text{ and } D_0 \subseteq D_1$.

- Fix $x \in 2^\mathbb{N}$. We want a descending sequence in $\mathcal{P}$

$$
(A_0, B_0, C_0, D_0) \geq (A_1, B_1, C_1, D_1) \geq \ldots
$$

such that $h(x)$ can be defined to be the unique element of

$$
\bigcap_{i}(A_i \cap B_i \cap C_i \cap D_i)
$$
Proof Sketch II

$h(x)$ will be the unique element of

$$\bigcap_{i}(A_i \cap B_i \cap C_i \cap D_i)$$

where the following is a descending sequence in $\mathbb{P}$

$$(A_0, B_0, C_0, D_0) \geq (A_1, B_1, C_1, D_1) \geq \ldots$$

Further

- $x \in \Gamma_5$ implies for some $i$ the sequence $D_i, D_{i+1}, \ldots$ is constant, and $C_i, C_{i+1}, \ldots$ generates a generic filter in $A_2 \upharpoonright D_i$.

- $x \notin \Gamma_5$ implies the sequence $D_0, D_1, \ldots$ generates a generic filter in $A_3$.

Hence $h^{-1}(E) = \Gamma_5$. 
Problem 1: Need $\bigcap_i (A_i \cap B_i \cap C_i \cap D_i) \neq \emptyset$

Lemma

Consider the Choquet game where each player plays elements of $\mathbb{P}$

<table>
<thead>
<tr>
<th></th>
<th>$p_0$</th>
<th>$p_2$</th>
<th>$p_4$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

such that $p_0 > p_1 > p_2 > \ldots$ and II wins if $\bigcap_i (A_i \cap B_i \cap C_i \cap D_i)$ is a singleton (where $p_i = (A_i, B_i, C_i, D_i)$).

Then II has a winning strategy in this game where $D_{2i+1} = D_{2i}$ and $C_{2i+1} = C_{2i}$. 
Problem 2: Cannot continuously build an interesting decreasing sequence

We can continuously build a sequence in $\mathbb{P}$:

$$(A_0, B_0, C_0, D_0), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), \ldots$$

such that:

- $A_{n+1} \subseteq A_n$ (so $h$ will be continuous).
- The sequence contains a subsequence that is decreasing in $\mathbb{P}$ and has the desired properties.

We make use of a technique known as “true stages” approximation. Essentially, true stages uses approximations to the jumps of $x$ to determine how elements of the above sequence should relate to one another.
We are continuously building a sequence in $\mathbb{P}$:

$$(A_0, B_0, C_0, D_0), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), \ldots$$

Each $s \in \mathbb{N}$ has an approximation to $x'$, $x''$ and $x'''$. If $s < t$ then

- If $t$’s approximation to $x'$ agrees with $s$’s then: $(B_t \subseteq B_s)$.
- If $t$’s approximation to $x''$ agrees with $s$’s then: $(C_t \subseteq C_s)$.
- If $t$’s approximation to $x'''$ agrees with $s$’s then: $(D_t \subseteq D_s)$. 
To apply true stages at stage $s$ of the construction you are given $s_3 \leq s_2 \leq s_1 \leq s_0 = s - 1$ and the following relationships:

\[
(A_{s_3}, B_{s_3}, C_{s_3}, D_{s_3}) 
\cup \ 
(A_{s_2}, B_{s_2}, C_{s_2}, D_{s_2}) 
\cup \ 
(A_{s_1}, B_{s_1}, C_{s_1}, D_{s_1}) 
\cup \ 
(A_{s_0}, B_{s_0}, C_{s_0}, D_{s_0})
\]

You must then make

\[
(A_s, B_s, C_s, D_s) < (A_{s_0}, B_{s_1}, C_{s_2}, D_{s_3})
\]

But what if $A_{s_0} \cap B_{s_1} \cap C_{s_2} \cap D_{s_3} = \emptyset$?
Use conditions \((A, B, C, D)\) such that:

- If \(A^* \in A_0\) and \(A^* \cap B \neq \emptyset\) then \(A^* \cap B \cap C \neq \emptyset\).
- If \(B^* \in A_2\) and \(B^* \cap C \neq \emptyset\) then \(B^* \cap C \cap D \neq \emptyset\).

Need to enrich \(A_1\) and \(A_2\).

1. For all \(B \in A_1\) and \(C \in A_2\) we want
   \[
   B \setminus \bigcup\{A \in A_0 : A \cap C = \emptyset\} \in A_1.
   \]

2. For all \(C \in A_2\) and \(D \in A_3\) we want
   \[
   C \setminus \bigcup\{B \in A_1 : B \cap D = \emptyset\} \in A_2.
   \]

Say \(A_0, A_1, A_2, A_3\) is densely closed if conditions 1) and 2) hold.
This is a difficult closure condition to achieve.
Decomposability

Let \( f : \mathcal{X} \to \mathcal{Y} \) be a function between Polish spaces. We will write

\[
\text{if for any } E \subseteq \mathcal{Y} \text{ that is } \Sigma^0_m \text{ we have that } f^{-1}(E) \text{ is } \Sigma^0_n.
\]

\[
\bullet f^{-1}(\Sigma^0_1) \subseteq \Sigma^0_1 \text{ is the same as } f \text{ being continuous.}
\]

\[
\bullet f^{-1}(\Sigma^0_1) \subseteq \Sigma^0_{n+1} \text{ is the same as } f \text{ being Baire Class } n.
\]
Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. We will write

$$f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$$

if for any $E \subseteq \mathcal{Y}$ that is $\Sigma^0_m$ we have that $f^{-1}(E)$ is $\Sigma^0_n$.

- $f^{-1}(\Sigma^0_1) \subseteq \Sigma^0_1$ is the same as $f$ being continuous.
- $f^{-1}(\Sigma^0_1) \subseteq \Sigma^0_{n+1}$ is the same as $f$ being Baire Class $n$.

### Decomposability Conjecture

Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\Delta^0_n$ sets on which $f$ is Baire Class $(n - m)$
Easy Direction

Decomposability Conjecture

Let \( f : \mathcal{X} \to \mathcal{Y} \) be a function between Polish spaces. Let \( n \geq m > 1 \) then the following are equivalent:

1. \( f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n \)

2. There is a countable partition of \( \mathcal{X} \) into \( \Delta^0_n \) sets on which \( f \) is Baire Class \( (n - m) \)

Let \( \{D_i\} \) be a partition of \( \mathcal{X} \) into \( \Delta^0_n \) sets on which \( f \) is Baire Class \( (n - m) \).

- Let \( U \) be a \( \Sigma^0_m \) subset of \( \mathcal{Y} \). For each \( i \) there a set \( E_i \) that is \( \Sigma^0_{m+n-m} \) and \( f^{-1}(U) \cap D_i = E_i \cap D_i \).
- Hence, \( f^{-1}(U) = \bigcup_i D_i \cap E_i \) which is \( \Sigma^0_n \).
- Thus \( f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n \).
## Results so far

### Decomposability Conjecture

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\Sigma_m^0) \subseteq \Sigma_n^0$
2. There is a countable partition of $\mathcal{X}$ into $\Delta_n^0$ sets on which $f$ is Baire Class $(n - m)$

- (1982) Case $n = m = 2$ proved (Jayne-Rodgers)
- (2009) Case $n = 3$, $m \in \{2, 3\}$ for $\mathcal{X} = \omega^\omega$ (Semmes)
- (2013-2016) 1. implies 2. with $\Delta_{n+1}^0$ sets. (Kihara, Gregoriades-Kihara-Ng)
- (2017) Case $n = 3$, $m \in \{2, 3\}$ for $\mathcal{X}$ Polish (Ding-Kihara-Semmes-Zhao)
Important Cases

Decomposability Conjecture

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$

2. There is a countable partition of $\mathcal{X}$ into $\Delta^0_n$ sets on which $f$ is Baire Class $(n - m)$

Proposition (Gregoriades-Kihara-Ng)

Fix $n > 1$. If the decomposability conjecture is true for this $n$ and $m = 2$, then the decomposability conjecture is true for this $n$ and all $m$. 
A Strategy

Let \( f : X \rightarrow Y \) be a function between Polish spaces such that \( X \) cannot be partitioned into countably many \( \Delta_n^0 \) sets on which \( f \) is not Baire Class \( n - 2 \).

- How do we show that there is a \( \Sigma_2^0 \) set \( E \subseteq Y \) such that \( f^{-1}(E) \) is not \( \Sigma_n^0 \)?

- Reduce the problem to the \( n = 3 \) and \( m = 2 \) case using:
  - A change of topology argument.
  - The earlier completeness criterion.
Change of Topology

Lemma

Given:

1. \( f : (\mathcal{X}, \tau) \to \mathcal{Y} \).
2. \( \tau' \) a topology on \( \mathcal{X} \) refining \( \tau \) and generated by \( \Pi_{n-3}^0 \) subsets of \( (\mathcal{X}, \tau) \).

If there is a partition \( \{D_i\}_{i \in \mathbb{N}} \) of \( (\mathcal{X}, \tau') \) into \( \Delta_3^0 \) sets on which \( f \) is Baire Class 1 with respect to \( \tau' \).

Then \( \{D_i\}_{i \in \mathbb{N}} \) is a \( \Delta_n^0 \) partition of \( (\mathcal{X}, \tau) \) on which \( f \) is Baire Class \( n - 2 \) with respect to \( \tau \).

Proof.

A \( \Sigma_m^0 \) or \( \Pi_m^0 \) set in \( (\mathcal{X}, \tau') \) is, respectively, \( \Sigma_{m+n-3}^0 \) or \( \Pi_{m+n-3}^0 \) in \( (\mathcal{X}, \tau) \).
An Overly Hopeful Approach

Goal Start with \( f : (\mathcal{X}, \tau) \to \mathcal{Y} \) that is not Baire Class \( n - 2 \) on any countable \( \Delta^0_n \) partition of \((\mathcal{X}, \tau)\).

1. Let \( A_0, A_1, \ldots, A_{n-3} \) be a densely closed, suitable sequence. Let \( \tau' \) be a new topology on \( \mathcal{X} \) obtained by making all elements of the sequence open.

2. Run the proof for the \( f^{-1}(\Sigma^0_2) \subseteq \Sigma^0_3 \) case to obtain a closed \( E \subseteq \mathcal{Y} \) and \( C \) a collection of closed sets such that:
   - \( \vdash_C f^{-1}(E) \)
   - \( \vdash A_{n-3} \upharpoonright_C f^{-1}((\mathcal{Y} \setminus E)) \) for any \( C \in C \).

3. Have \( A_1, A_2, \ldots, A_{n-3}, A_{n-2} = C \) a densely closed, suitable sequence such that:
   - \( \vdash_C f^{-1}(E) \)
   - \( \vdash A_{n-3} \upharpoonright_C f^{-1}((\mathcal{Y} \setminus E)) \) for any \( C \in C \).

By criterion \( f^{-1}(E) \) is \( \Pi^0_n \)-hard so \( f^{-1}(\Sigma^0_2) \subseteq \Sigma^0_n \) is false.
Problem

- The main problem is getting $A_1, A_2, \ldots, A_{n-3}, A_{n-2} = C$ a densely closed, suitable sequence.

- We can iterate our attempts up through the countable ordinals as follows:
  - Start with some $A_0, A_1, \ldots, A_{n-3}$
  - Change topology using this sequence.
  - Obtain $C$.
  - If $A_0, A_1, \ldots, A_{n-3}, A_{n-2} = C$ is not densely closed and suitable, then enrich $A_0, A_1, \ldots, A_{n-3}$.
  - Repeat above at successor ordinals.
  - Take unions at limits.

**Question** Is there a countable closure point?
Harrington’s Theorem

Theorem (Harrington)
Assume the full axiom of determinacy. There is no 1-1 $\aleph_1$ sequence of Borel sets all of fixed rank in the Borel hierarchy.

- It is not difficult to modify Harrington’s theorem to show that under AD there is no $\aleph_1$ length sequence of increasing countable collections of Borel sets of a fixed rank.

- Hence we get a closure point if we can run our approach in ZF+AD+DC.
The $n = 3$ $m = 2$ Case

Theorem

(ZF+DC) Fix $f : \mathcal{X} \to \mathcal{Y}$ there is a function $\Phi$ with domain the set of Polish topologies on $\mathcal{X}$ such that if $f : (\mathcal{X}, \tau)$ is not Baire class 1 on a countable $\Delta^0_3$ partition of $\mathcal{X}$ then $\Phi(\tau)$ is either:

- $(C, D)$ with $D \subseteq \mathcal{Y}$ where $C$ witnesses that $f^{-1}(D)$ is $\Pi^0_3$ hard.
- $y \in \mathcal{Y}$ such that $f^{-1}(\{y\})$ is $\Pi^0_3$ hard.
- $B \subseteq \mathcal{Y}$ a finite boolean combination of basic open sets such that $f^{-1}(B)$ is $\Pi^0_3$ hard.

Theorem

Under AD+DC, the decomposability conjecture is true for $n > 2$ and $m = 2$. 
How Much Determinacy?

How to apply Harrington’s theorem to 2nd order arithmetic.

- Harrington’s Theorem uses a game where (among other things) I and II both play countable linear orders and win if they play a well-order and their opponent does not.

- Thus need determinacy of the difference of analytic sets.

- The $\aleph_1$ length sequence of increasing countable collections of Borel sets of a fixed rank can be replace by a mapping of well-orderings to countable collections of Borel sets.

- **BUT** for this to work the topological arguments must be independent of the presentation of the topology on $X$. (Otherwise things break down at limit stages.)
Conclusion

**Theorem**

Assume $\Pi^1_2$ determinacy. Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\Sigma^0_m) \subseteq \Sigma^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\Delta^0_n$ sets on which $f$ is Baire Class $(n - m)$

- We believe that in fact $\sigma(\Sigma^1_1)$ determinacy suffices.
- We are hopefully that the use of determinacy can be removed.
Thanks for your attention.