Completeness for Σ^0_{n+2} Sets

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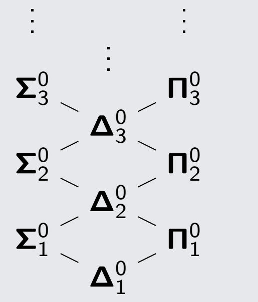
Outline

Borel Hierarchy

- Let $\mathcal{X}$ be an uncountable Polish space and $E \subseteq \mathcal{X}$.
- The Borel classes are closed under continuous preimages.
- How can you show that E is $\mathbf{\Sigma}^0_n$-complete?
- Give a $\mathbf{\Sigma}^0_n$ definition of E and show a complete $\mathbf{\Sigma}^0_n$ set is continuously reducible to it.

Borel Hierarchy



Some Complete Sets

Work in $2^{\mathbb{N}}$. Define:

- $\Gamma_1 = \{x \colon \exists n(x(n) = 1)\}$.
- Γ_{i+1} to be $\{x \in 2^{\mathbb{N}} \colon \exists n(x^{[n]} \notin \Gamma_i)\}$.

(where $x^{[n]}(i) = x(\langle n, i \rangle))$.

- It easy to show that Γ_i is $\mathbf{\Sigma}^0_i$ complete.
- For example, Γ_3 is

$$\{x \colon \exists i \, \forall j \, \exists k (x(\langle i, \langle j, k \rangle\rangle) = 1)\}$$

Showing Γ_i is Complete

Completeness for Γ_3

Let $E \subseteq 2^{\mathbb{N}}$ be $\boldsymbol{\Sigma}^0_3$. Hence

$$E = \bigcup_i \bigcap_j \bigcup_k D_{\langle i,j,k \rangle}$$

where each $D_{\langle i,j,k \rangle}$ is basic clopen.

Define $h : \mathcal{X} \to 2^{\mathbb{N}}$ by

$$h(x)(\langle i,j,k \rangle) = \begin{cases} 1 & x \in D_{\langle i,j,k \rangle} \\ 0 & \text{otherwise.} \end{cases}$$

Then h continuously reduces E to Γ_3, so Γ_3 is $\boldsymbol{\Sigma}^0_3$-complete.

But how about a set that isn't as nice as Γ_i?

Orthogonal Forcings

Idea: Fix $E \subseteq \mathcal{X}$. Construct $f : 2^{\mathbb{N}} \to \mathcal{X}$.

Use forcing with a countable collection of closed sets, and Cohen forcing to construct the reduction $f : 2^{\mathbb{N}} \to \mathcal{X}$.

- Use Cohen forcing to obtain $f(x) \in E$.
- Use the closed set forcing to obtain $f(x) \notin E$.

Orthogonal Forcings

- Let $\mathcal{A}_0$ be all basic open subsets of $\mathcal{X}$.
- Let $\mathcal{A}_1$ be a countable collection of closed subsets of $\mathcal{X}$ that is
  - closed under finite intersection.
  - contains the closures of all basic open sets.
- Order $\mathcal{A}_1$ under inclusion
- A generic filter in this partial order uniquely determines an element of $\mathcal{X}$.
- Write $\Vdash_{\mathcal{A}_1} B$ if every $\mathcal{A}_1$-generic element of $\mathcal{X}$ is a member of $B \subseteq \mathcal{X}$.
- The set of these generic elements of $\mathcal{X}$ is $\mathbf{\Pi}^0_3$.

Cohen forcing inside elements of $\mathcal{A}_1$.

- Let $C \in \mathcal{A}_1$.
- Define $\mathcal{A}_0 \restriction_C = \{B \in \mathcal{A}_0 \colon B \cap C \neq \emptyset\}$.
- Define an ordering on $\mathcal{A}_0 \restriction_C$ by $B_0 \leq B_1$ if $B_0 \cap C \supseteq B_1 \cap C$.
- A generic filter in $\mathcal{A}_0 \restriction_C$ uniquely determines an element of C.
- Write $\Vdash_{\mathcal{A}_0 \restriction_C} B$ if every such element of C is a member of $B \subseteq \mathcal{X}$.
- The set of generics for $\mathcal{A}_0 \restriction_C$ for all $C \in \mathcal{A}_1$ is $\mathbf{\Sigma}^0_3$.

Completeness for $\mathbf{\Sigma}^0_3$ Sets

Theorem (Abstraction of a folklore technique)

Let E be a subset of $\mathcal{X}$ a Polish space. Let $\mathcal{A}_1$ be a collection of closed subsets of $\mathcal{X}$ that is: closed under finite intersections, and contains the closures of all basic open sets. If

- $\Vdash_{\mathcal{A}_1} \mathcal{X} \setminus E$.
- For all $C \in \mathcal{A}_1$, $\Vdash_{A_0 \restriction_C} E$

Then E is $\mathbf{\Sigma}^0_3$ complete.

Proof Idea

Define a reduction $h : 2^\omega \to \mathcal{X}$ such that $f^{-1}(E) = \{x \in 2^\omega : \exists n(x^{[n]} \text{ is infinite}).\}$.

Completeness for $\mathbf{\Sigma}^0_3$ Sets

Fix x and define $p^x_1, p^x_2 \ldots.$

- Assume $p^x_n = ((C_0, C_1, \ldots, C_k), B)$ where each $C_i \in \mathcal{A}_1$ with $C_0 \supseteq C_1 \supseteq \ldots \supseteq C_k$, $B \in \mathcal{A}_0$, radius of B is less than 2^{-n}, and $B \cap C_k \neq \emptyset$.
- If $x(n) = 0$ then define p^x_{n+1} map x to be

$$((C_0, C_1, \ldots, C_k, C_{k+1}), \hat{B})$$

  where C_{k+1} meets the next dense open set inside $\mathcal{A}_1$ and $\hat{B} \subseteq B$.
- If $x(n) = 1$ and $n = \langle e, m \rangle$ for some m with $e \leq k$, then define p^x_{n+1} to be

$$((C_0, C_1, \ldots, C_e), \hat{B})$$

  where $\hat{B} \subseteq B_n$ that meets the next dense set inside $\mathcal{A}_0 \upharpoonright_{C_e}$
- Define $h(x)$ to be unique element of $\bigcap_n \pi_2(p^x_n)$.

A Completeness Criterion for $\mathbf{\Sigma}^0_n$ Sets

Theorem

Let $\mathcal{X}$ be a Polish space. Let $n \geq 3$. A set $E \subseteq \mathcal{X}$ is $\mathbf{\Sigma}^0_n$ hard if and only if there is "densely closed" "suitable sequence" $\mathcal{A}_0, \ldots, \mathcal{A}_{n-2}$ such that:

1. $\Vdash_{\mathcal{A}_{n-2}} \mathcal{X} \setminus E$.
2. $\Vdash_{\mathcal{A}_{n-3}\restriction_B} E$ for all $B \in \mathcal{A}_{n-2}$.

Suitable sequence means:

- $\mathcal{A}_0$ is all basic open sets.
- $\mathcal{A}_i$ is a countable collection of $\mathbf{\Pi}^0_i$ sets.
- $\mathcal{A}_i$ is closed under finite intersection (with itself and appropriate lower $\mathcal{A}_j$'s).
- $\mathcal{A}_{i+2}$ is expressible in terms of $\mathcal{A}_i$.

A Completeness Criterion for $\mathbf{\Sigma}^0_n$ Sets

Theorem

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1. $\Vdash_{\mathcal{A}_{n-2}} \mathcal{X} \setminus E$.
2. $\Vdash_{\mathcal{A}_{n-3}\restriction_B} E$ for all $B \in \mathcal{A}_{n-2}$.

1. means any generic element obtained by forcing with $\mathcal{A}_{n-2}$ is in $\mathcal{X} \setminus E$.
2. means for any $B \in \mathcal{A}_{n-2}$, any generic element obtained by forcing with $\{C \in \mathcal{A}_{n-3} \colon C \cap B \neq \emptyset\}$ is in E.

Proof Idea

Easy direction:

- Show that Γ_n has these properties.
- If E is $\mathbf{\Sigma}^0_n$ hard find $f : 2^{\mathbb{N}} \to \mathcal{X}$ such that $f^{-1}(E) = \Gamma_n$.
- Use f to push forward these properties to $\mathcal{X}$.

Hard direction:

- Run the $\mathbf{\Sigma}^0_3$ construction on top of a "true-stages" argument.
- The case $n = 5$ is representative.

Assume we have a densely closed, suitable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that:

- $\Vdash_{\mathcal{A}_3} \mathcal{X} \setminus E$.
- $\Vdash_{\mathcal{A}_2 \restriction_B} E$ for all $B \in \mathcal{A}_3$.

Proof Sketch I

To show E is $\mathbf{\Sigma}^0_5$-hard will build a continuous function $h : 2^{\mathbb{N}} \to \mathcal{X}$ such that $h^{-1}(E) = \Gamma_5$.

- $\mathbb{P} = \{(A, B, C, D) \in \mathcal{A}_0 \times \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \colon A \cap B \cap C \cap D \neq \emptyset\}$.
- Order $\mathbb{P}$ by $(A_0, B_0, C_0, D_0) \leq (A_1, B_1, C_1, D_1)$ if
$$A_0 \subseteq A_1, B_0 \subseteq B_1, C_0 \subseteq C_1, \text{ and } D_0 \subseteq D_1.$$
- Fix $x \in 2^{\mathbb{N}}$. We want a descending sequence in $\mathbb{P}$
$$(A_0, B_0, C_0, D_0) \geq (A_1, B_1, C_1, D_1) \geq \dots$$
such that $h(x)$ can be defined to be the unique element of
$$\bigcap_i (A_i \cap B_i \cap C_i \cap D_i)$$

Proof Sketch II

$h(x)$ will be the unique element of

$$\bigcap_i (A_i \cap B_i \cap C_i \cap D_i)$$

where the following is a descending sequence in $\mathbb{P}$

$$(A_0, B_0, C_0, D_0) \geq (A_1, B_1, C_1, D_1) \geq \dots$$

Further

- $x \in \Gamma_5$ implies for some i the sequence $D_i, D_{i+1}, \dots$ is constant, and $C_i, C_{i+1}, \dots$ generates a generic filter in $\mathcal{A}_2 \restriction_{D_i}$.
- $x \notin \Gamma_5$ implies the sequence $D_0, D_1, \dots$ generates a generic filter in $\mathcal{A}_3$.

Hence $h^{-1}(E) = \Gamma_5$.

Problem 1: Need $\bigcap_i (A_i \cap B_i \cap C_i \cap D_i) \neq \emptyset$

Lemma

Consider the Choquet game where each player plays elements of $\mathbb{P}$

I	p_0		p_2		p_4		$\dots$
II		p_1		p_3		p_5	$\dots$

such that $p_0 > p_1 > p_2 > \dots$ and II wins if $\bigcap_i (A_i \cap B_i \cap C_i \cap D_i)$ is a singleton (where $p_i = (A_i, B_i, C_i, D_i)$).

Then II has a winning strategy in this game where $D_{2i+1} = D_{2i}$ and $C_{2i+1} = C_{2i}$.

Problem 2: Cannot continuously build an interesting decreasing sequence

We can continuously build a sequence in $\mathbb{P}$:

$$(A_0, B_0, C_0, D_0), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), \ldots$$

such that:

- $A_{n+1} \subseteq A_n$ (so h will be continuous).
- The sequence contains a subsequence that is decreasing in $\mathbb{P}$ and has the desired properties.

We make use of a technique known as "true stages" approximation.

Essentially, true stages uses approximations to the jumps of x to determine how elements of the above sequence should relate to one another.

True Stages Contract

We are continuously building a sequence in $\mathbb{P}$:

$$(A_0, B_0, C_0, D_0), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), \ldots$$

Each $s \in \mathbb{N}$ has an approximation to x', x'' and x'''. If $s < t$ then

- If t's approximation to x' agrees with s's then: $(B_t \subseteq B_s)$.
- If t's approximation to x'' agrees with s's then: $(C_t \subseteq C_s)$.
- If t's approximation to x''' agrees with s's then: $(D_t \subseteq D_s)$.

Key True Stages Lemma

To apply true stages at stage s of the construction you are given $s_3 \leq s_2 \leq s_1 \leq s_0 = s - 1$ and the following relationships:

$$\begin{array}{cccc}
(A_{s_3}, & B_{s_3}, & C_{s_3}, & D_{s_3}) \\
\cup & \cup & \cup & \cup \\
(A_{s_2}, & B_{s_2}, & C_{s_2}, & D_{s_2}) \\
\cup & \cup & \cup & \\
(A_{s_1}, & B_{s_1}, & C_{s_1}, & D_{s_1}) \\
\cup & \cup & & \\
(A_{s_0}, & B_{s_0}, & C_{s_0}, & D_{s_0})
\end{array}$$

You must then make

$$(A_s, B_s, C_s, D_s) < (A_{s_0}, B_{s_1}, C_{s_2}, D_{s_3})$$

But what if $A_{s_0} \cap B_{s_1} \cap C_{s_2} \cap D_{s_3} = \emptyset$?

Solution

Use conditions (A, B, C, D) such that:

- If $A^* \in \mathcal{A}_0$ and $A^* \cap B \neq \emptyset$ then $A^* \cap B \cap C \neq \emptyset$.
- If $B^* \in \mathcal{A}_2$ and $B^* \cap C \neq \emptyset$ then $B^* \cap C \cap D \neq \emptyset$.

Need to enrich $\mathcal{A}_1$ and $\mathcal{A}_2$.

1. For all $B \in \mathcal{A}_1$ and $C \in \mathcal{A}_2$ we want
$$B \setminus \bigcup\{A \in \mathcal{A}_0 \colon A \cap C = \emptyset\} \in \mathcal{A}_1.$$
2. For all $C \in \mathcal{A}_2$ and $D \in \mathcal{A}_3$ we want
$$C \setminus \bigcup\{B \in \mathcal{A}_1 \colon B \cap D = \emptyset\} \in \mathcal{A}_2.$$

Say $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ is densely closed if conditions 1) and 2) hold.

This is a difficult closure condition to achieve.

Decomposability

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. We will write

$$f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$$

if for any $E \subseteq \mathcal{Y}$ that is $\mathbf{\Sigma}^0_m$ we have that $f^{-1}(E)$ is $\mathbf{\Sigma}^0_n$.

- $f^{-1}(\mathbf{\Sigma}^0_1) \subseteq \mathbf{\Sigma}^0_1$ is the same as f being continuous.
- $f^{-1}(\mathbf{\Sigma}^0_1) \subseteq \mathbf{\Sigma}^0_{n+1}$ is the same as f being Baire Class n.

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Decomposability Conjecture

Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n - m)$

Easy Direction

Decomposability Conjecture

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n-m)$

Let $\{D_i\}$ be a partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n-m)$.

- Let U be a $\mathbf{\Sigma}^0_m$ subset of $\mathcal{Y}$. For each i there a set E_i that is $\mathbf{\Sigma}^0_{m+n-m}$ and $f^{-1}(U) \cap D_i = E_i \cap D_i$.
- Hence, $f^{-1}(U) = \bigcup_i D_i \cap E_i$ which is $\mathbf{\Sigma}^0_n$.
- Thus $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$.

Results so far

Decomposability Conjecture

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n-m)$

- (1982) Case $n = m = 2$ proved (Jayne-Rodgers)
- (2009) Case $n = 3$, $m \in \{2,3\}$ for $\mathcal{X} = \omega^\omega$ (Semmes)
- (2013-2016) 1. implies 2. with $\mathbf{\Delta}^0_{n+1}$ sets. (Kihara, Gregoriades-Kihara-Ng)
- (2017) Case $n = 3$, $m \in \{2,3\}$ for $\mathcal{X}$ Polish (Ding-Kihara-Semmes-Zhao)

Important Cases

Decomposability Conjecture

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n - m)$

Proposition (Gregoriades-Kihara-Ng)

Fix $n > 1$. If the decomposability conjecture is true for this n and $m = 2$, then the decomposability conjecture is true for this n and all m.

A Strategy

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces such that $\mathcal{X}$ cannot be partitioned into countably many $\mathbf{\Delta}^0_n$ sets on which f is not Baire Class $n-2$.

- How do we show that there is a $\mathbf{\Sigma}^0_2$ set $E \subseteq \mathcal{Y}$ such that $f^{-1}(E)$ is not $\mathbf{\Sigma}^0_n$?
- Reduce the problem to the $n = 3$ and $m = 2$ case using:
  - A change of topology argument.
  - The earlier completeness criterion.

Change of Topology

Lemma

Given:

- $f : (\mathcal{X}, \tau) \to \mathcal{Y}$.
- τ' a topology on $\mathcal{X}$ refining τ and generated by $\mathbf{\Pi}^0_{n-3}$ subsets of $(\mathcal{X}, \tau)$.

If there is a partition $\{D_i\}_{i \in \mathbb{N}}$ of (X, τ') into $\mathbf{\Delta}^0_3$ sets on which f is Baire Class 1 with respect to τ'.

Then $\{D_i\}_{i \in \mathbb{N}}$ is a $\mathbf{\Delta}^0_n$ partition of $(\mathcal{X}, \tau)$ on which f is Baire Class $n-2$ with respect to τ.

Proof.

A $\mathbf{\Sigma}^0_m$ or $\mathbf{\Pi}^0_m$ set in (X, τ') is, respectively, $\mathbf{\Sigma}^0_{m+n-3}$ or $\mathbf{\Pi}^0_{m+n-3}$ in $(\mathcal{X}, \tau)$. □

An Overly Hopeful Approach

Goal Start with $f : (\mathcal{X}, \tau) \to \mathcal{Y}$ that is not Baire Class $n-2$ on any countable $\mathbf{\Delta}^0_n$ partition of $(\mathcal{X}, \tau)$.

1. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-3}$ be a densely closed, suitable sequence. Let τ' be a new topology on $\mathcal{X}$ obtained by making all elements of the sequence open.
2. Run the proof for the $f^{-1}(\mathbf{\Sigma}^0_2) \subseteq \mathbf{\Sigma}^0_3$ case to obtain a closed $E \subseteq \mathcal{Y}$ and $\mathcal{C}$ a collection of closed sets such that:
   - $\Vdash_{\mathcal{C}} f^{-1}(E)$
   - $\Vdash_{\mathcal{A}_{n-\mathbf{3}} \restriction C} f^{-1}(\mathcal{Y} \setminus E)$ for any $C \in \mathcal{C}$.
3. Have $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n-3}, \mathcal{A}_{n-2} = \mathcal{C}$ a densely closed, suitable sequence such that:
   - $\Vdash_{\mathcal{C}} f^{-1}(E)$
   - $\Vdash_{\mathcal{A}_{n-\mathbf{3}} \restriction C} f^{-1}(\mathcal{Y} \setminus E)$ for any $C \in \mathcal{C}$.

By criterion $f^{-1}(E)$ is $\mathbf{\Pi}^0_n$-hard so $f^{-1}(\mathbf{\Sigma}^0_2) \subseteq \mathbf{\Sigma}^0_n$ is false.

Problem

- The main problem is getting $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n-3}, \mathcal{A}_{n-2} = \mathcal{C}$ a densely closed, suitable sequence.
- We can iterate our attempts up through the countable ordinals as follows:
  - Start with some $\mathcal{A}_0, \mathcal{A}_1, \ldots, A_{n-3}$
  - Change topology using this sequence.
  - Obtain $\mathcal{C}$.
  - If $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-3}, \mathcal{A}_{n-2} = \mathcal{C}$ is not densely closed and suitable, then enrich $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-3}$.
  - Repeat above at successor ordinals.
  - Take unions at limits.

Question Is there a countable closure point?

Harrington's Theorem

Theorem (Harrington)

Assume the full axiom of determinacy. There is no 1-1 $\aleph_1$ sequence of Borel sets all of fixed rank in the Borel hierarchy.

- It is not difficult to modify Harrington's theorem to show that under AD there is no $\aleph_1$ length sequence of increasing countable collections of Borel sets of a fixed rank.
- Hence we get a closure point if we can run our approach in ZF+AD+DC.

The $n = 3$ $m = 2$ Case

Theorem

(ZF+DC) Fix $f : \mathcal{X} \to \mathcal{Y}$ there is a fuction Φ with domain the set of Polish topologies on $\mathcal{X}$ such that if $f : (\mathcal{X}, \tau)$ is not Baire class 1 on a countable $\mathbf{\Delta}^0_3$ parition of $\mathcal{X}$ then $\Phi(\tau)$ is either:

- $(\mathcal{C}, D)$ with $D \subseteq \mathcal{Y}$ where $\mathcal{C}$ witnesses that $f^{-1}(D)$ is $\mathbf{\Pi}^0_3$ hard.
- $y \in Y$ such that $f^{-1}(\{y\})$ is $\mathbf{\Pi}^0_3$ hard.
- $B \subseteq \mathcal{Y}$ a finite boolean combination of basic open sets such that $f^{-1}(B)$ is $\mathbf{\Pi}^0_3$ hard.

Theorem

Under AD+DC, the decomposability conjecture is true for $n > 2$ and $m = 2$.

How Much Determinacy?

How to apply Harrington's theorem to 2nd order arithmetic.

- Harrington's Theorem uses a game where (among other things) I and II both play countable linear orders and win if they play a well-order and their opponent does not.

- Thus need determinacy of the difference of analytic sets.

- The $\aleph_1$ length sequence of increasing countable collections of Borel sets of a fixed rank can be replace by a mapping of well-orderings to countable collections of Borel sets.

- BUT for this to work the topological arguments must be independent of the presentation of the topology on $\mathcal{X}$. (Otherwise things break down at limit stages.)

Conclusion

Theorem

Assume Π^1_2 determinacy. Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between Polish spaces. Let $n \geq m > 1$ then the following are equivalent:

1. $f^{-1}(\mathbf{\Sigma}^0_m) \subseteq \mathbf{\Sigma}^0_n$
2. There is a countable partition of $\mathcal{X}$ into $\mathbf{\Delta}^0_n$ sets on which f is Baire Class $(n - m)$

- We believe that in fact $\sigma(\Sigma^1_1)$ determinacy suffices.
- We are hopefully that the use of determinacy can be removed.

Thank You

Thanks for your attention.