On Σ–preorderings in HF($\mathbb{R}$)

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HF($\mathbb{R}$): what is it?

$\mathbb{HF}(\mathbb{R})$: hereditarily finite superstructure over $\mathbb{R}$

Basic set: all the sets which can be explicitly written down using {, }, $\varnothing$, r ($r \in \mathbb{R}$).

Examples: $\varnothing$, $\{\varnothing, \sqrt{2}\}$, $\{7, \{\{\varnothing, 92\}, 3, \{\varnothing\}\}\}$, etc.

Σ–formulas: a specific class of formulas that define analogs of c.e. sets (we omit the definition).

One can also consider hereditarily finite superstructures $\mathbb{HF}(\mathfrak{M})$ for any structure $\mathfrak{M}$ of finite predicate signature.

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Why $\mathbb{HF}(\mathbb{R})$?

Definability by means of Σ–formulas over $\mathbb{HF}(\mathbb{R})$ can be viewed as "computable enumerability" in a high level programming language in which we have **exact** realizations of the field $\mathbb{R}$ of real numbers and in addition we can compute (and use in further computations) all the roots of polynomial equations from their coefficients.

Σ–presentability of structures

Σ–Presentable structures: analog of the notion of computable structures (c.e. is replaced with Σ–definability)

A presentation of an algebraic structure $\mathfrak{M}$ of a finite predicate signature is any assignment of codes from some $A \subseteq \mathbb{HF}(\mathbb{R})$ to its elements, i.e., a mapping $\nu : A \subseteq \mathbb{HF}(\mathbb{R}) \xrightarrow{onto} |\mathfrak{M}|$.

- If ν is 1–1 then ν is said to be *simple*.
- If $D(\mathfrak{M}, \nu)$ is Σ–definable with parameters in $\mathbb{HF}(\mathbb{R})$ then ν is a *Σ–presentation of* $\mathfrak{M}$ over $\mathbb{HF}(\mathbb{R})$.
- If $D^+(\mathfrak{M}, \nu)$ (the positive diagram) is Σ–definable with parameters in $\mathbb{HF}(\mathbb{R})$ then ν is said to be a *positive Σ–presentation of* $\mathfrak{M}$ over $\mathbb{HF}(\mathbb{R})$.

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Theorem (Yu.L. Ershov, 1985, 1995)

- *$\mathbb{R}$ and $\mathbb{C}$ have no Σ–presentations in a hereditarily finite superstructure over an infinite set.*
- *$\mathbb{C}$ has a Σ–presentation over any dense linearly ordered set of cardinality 2^ω.*
- *$\mathbb{R}$ has no Σ–presentations over such superstructures.*

Theorem (M. and M. Korovina)

- *If a structure has a Σ–presentation over $\mathbb{HF}(\mathbb{R})$ without parameters such that each its element has at most countable set of codes, then this structure has a computable copy.*
- *Without restrictions on the cardinalities of sets of codes for elements, there are structures of arbitrarily high hyperarithmetical complexity (still without parameters!).*
- *If a countable structure has a Σ–presentation over $\mathbb{HF}(\mathbb{R})$ then it has an isomorphic hyperarithmetical copy.*
- *Each at most countable structure has a Σ–presentation over $\mathbb{HF}(\mathbb{R})$ with at most one parameter.*

Model existence theorem

Theorem (M.)

Any countable consistent theory with infinite models has a model of cardinality 2^ω which is Σ–definable over $\mathbb{HF}(\mathbb{R})$.

(M.) Some structures **without** simple Σ–presentations over $\mathbb{HF}(\mathbb{R})$:

- the Boolean algebra of all subsets of ω and its quotient modulo the ideal of finite sets
- the group of all permutations on ω and its quotient modulo the subgroup of all finitary permutations
- the semigroup of all mappings from ω to ω
- the lattices of all open and all closed subsets of the reals
- the group of all permutations of $\mathbb{R}$ Σ–definable over $\mathbb{HF}(\mathbb{R})$
- the semigroup of all such mappings from $\mathbb{R}$ to $\mathbb{R}$
- the semigroup of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- some structures of nonstandard analysis (including ultrapowers of $\mathbb{R}$ modulo Fréchet ultrafilter with distinguished infinitesimal and standard elements)

Number of Σ–presentations

Theorem

There exist 2^ω pairwise non Σ–isomorphic presentations of the natural ordering $<$ on $\mathbb{R}$.

The basic result

Theorem

Suppose that $\preccurlyeq$ and L are subsets of $\mathbb{HF}(\mathbb{R})$ definable by means of Σ–formulas with parameters and $\preccurlyeq$ is a preordering on L. Then there is no isomorphic embedding from ω_1 into $\langle L; \preccurlyeq\rangle$.

Remark The above result fails to be true for Borel preorders: $\langle P(\omega); \subseteq^*\rangle$ can serve as a counterexample.

Idea of the proof (1):

Definition

$\mathrm{sp}(a)$: *support of a, the set of all the reals that are 'mentioned' in a.*

Examples:
$\mathrm{sp}(\varnothing) = \varnothing$, $\mathrm{sp}(\{1, \{1\}\}) = \{1\}$,
$\mathrm{sp}(\{0, \{1, 2, \{\sqrt{3}\}\}\}) = \{0, 1, 2, \sqrt{3}\}$
etc.

Definition

Let $\bar{p} \in \mathbb{R}^{<\omega}$ and $a \in \mathbb{HF}(\mathbb{R})$. The $\bar{p}$–dimension of a ($\dim_{\bar{p}}(a)$) is the cardinality of maximal algebraically independent subset of $\mathrm{sp}(a)$ over the field $\mathbb{R}(\bar{p})$.

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Idea of the proof (2):

Assume L and $\preccurlyeq$ are definable by Σ–formulas with parameters $\bar{p}$, $\preccurlyeq$ is a preordering on L and $A \subseteq L$ has the property $\langle A; \preccurlyeq\rangle \cong \omega_1$. We can assume that all the elements of A has the same dimension and this dimension n_0 is the minimal possible.

Lemma

$\forall x \in A \exists y \in L \exists z \in A(x \preccurlyeq y \preccurlyeq z \wedge \neg(z \preccurlyeq x) \wedge \dim_{\bar{p}}(y) < n_0).$

(we can always make a step aside to get a smaller dimension!)

Idea of the proof (3):

Ideas and facts used in the proof of Lemma:

- If $X \subseteq \mathbb{HF}(\mathbb{R})$ is countable then $D = \{a \mid \mathrm{sp}(a) \subseteq c\ell_{\bar{p}}(X)\}$ is countable. It follows that if S is an ω_1–chain then there is a $b \in S$ such that there are no elements of D greater than b.
- (Algebraic generalization principle) If φ is a Σ–formula, $\bar{a}$ is algebraically independent over $\bar{p}$, and $\mathbb{HF}(\mathbb{R}) \models \varphi(\bar{p}, \bar{a})$ then $\varphi(\bar{p}, \bar{x})$ is true in some open neighborhood of $\bar{a}$.
- If F, G, H are Σ–functions, $F(\bar{p}, \bar{a}) \preccurlyeq G(\bar{p}, \bar{b}, c) \preccurlyeq H(\bar{p}, \bar{d})$, and c is algebraically independent over $\bar{p}, \bar{a}, \bar{b}, \bar{d}$ then for some rational $r \in \mathbb{Q}$ it is true that $F(\bar{p}, \bar{a}) \preccurlyeq G(\bar{p}, \bar{b}, r) \preccurlyeq H(\bar{p}, \bar{d})$. Thus, $G(\bar{p}, \bar{b}, r)$ becomes a smaller $\bar{p}$–dimension.

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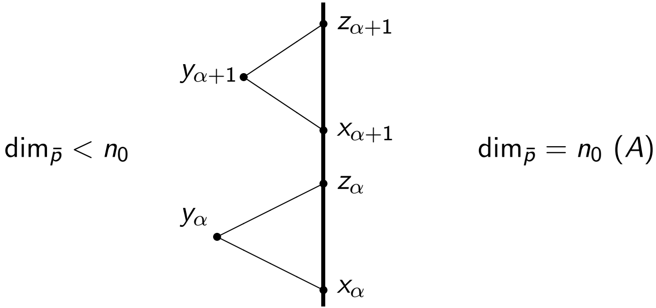
Idea of the proof (4):

Define elements $x_\alpha,\ z_\alpha \in A$, $y_\alpha \in L$, $\alpha < \omega_1$ by induction.

x_α: any element from A that strictly majorates $\{z_\gamma \mid \gamma < \alpha\}$.
Select $y_\alpha \in L,\ z_\alpha \in A$ so that $\dim_{\bar{p}}(y_\alpha) < n_0$, $\neg(z_\alpha \preccurlyeq x_\alpha)$, and $x_\alpha \preccurlyeq y_\alpha \preccurlyeq z_\alpha$:



Idea of the proof (5):

Lemma

$\alpha < \beta < \omega_1 \;\Leftrightarrow\; y_\alpha \prec y_\beta$

It follows that $\alpha \mapsto y_\alpha$ is an isomorphic embedding from ω_1 into $\langle L; \preccurlyeq \rangle$ such that all the $\bar{p}$–dimensions of y_α, $\alpha < \omega$ are strictly less than n_0.

And it easily follows that n_0 is not minimal possible!

Presentability of ordinals

Corollary

For any ordinal α the following conditions are equivalent:

1. α *has a simple* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
2. α *has a* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
3. α *has a positive* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
4. $\alpha < \omega_1$.

Presentability of ordinals without parameters

Corollary

For any ordinal α the following conditions are equivalent:

1. α *has a simple* Σ*–presentation without parameters over* $\mathbb{HF}(\mathbb{R})$
2. α *has a* Σ*–presentation without parameters over* $\mathbb{HF}(\mathbb{R})$
3. $\alpha < \omega_1^{CK}$

Presentability of Gödel constructive sets

$$L_0 = \varnothing, \quad L_{\alpha+1} = \mathtt{Def}\,(L_\alpha), \quad L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha, \text{ for limit } \gamma$$

Corollary

For any ordinal α the following conditions are equivalent:

1. $\langle L_\alpha; \in\rangle$ *has a simple Σ–presentation over* $\mathbb{HF}(\mathbb{R})$
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Presentability of Gödel constructive sets without parameters

Theorem

For any ordinal α the following conditions are equivalent:

1. *The structure $\langle L_\alpha; \in\rangle$ has a simple Σ–presentation over $\mathbb{HF}(\mathbb{R})$ without parameters*
2. $\alpha \leqslant \omega$.

(Here we don't need the basic theorem)

Corollary

Assume that $\langle L; \leqslant \rangle$ is an arbitrary partially ordered set in which for any at most countable chain $C \subseteq L$ there exists an $x \in L \setminus C$ with the property $C \leqslant x$.
Then $\langle L; \leqslant \rangle$ has no positive Σ–presentations over $\mathbb{HF}(\mathbb{R})$ with parameters (it follows that it has no neither Σ–presentations nor simple Σ–presentations with parameters).

Nonpresentability of some degree structures

Theorem

The partially ordered sets of Turing, m–, 1–, and tt–degrees have no positive Σ–presentations over $\mathbb{HF}(\mathbb{R})$ with parameters. (It follows that they have no neither Σ–presentations nor simple Σ–presentations with parameters).

Presentability over $\mathbb{C}$

Corollary

Let α be an ordinal. Then the following conditions are equivalent:

1. α *has a simple Σ–presentation over* $\mathbb{HF}(\mathbb{C})$
2. α *has a Σ–presentation over* $\mathbb{HF}(\mathbb{C})$
3. α *has a positive Σ–presentation over* $\mathbb{HF}(\mathbb{C})$
4. $\alpha < \omega_1^{CK}$

Contents of the talk

- Early studies
- Non–embeddability of ω_1 into Σ–definable preorderings over $\mathbb{HF}(\mathbb{R})$ (basic result)
- Descriptions of Σ–presentable ordinals (with parameters and without them) over $\mathbb{HF}(\mathbb{R})$
- Description of Σ–presentable Gödel constructive sets (with parameters and without them) over $\mathbb{HF}(\mathbb{R})$
- Non–Σ–presentability of some degree structures (T–, m–, 1–, tt–) over $\mathbb{HF}(\mathbb{R})$
- Description of Σ–presentable ordinals over $\mathbb{HF}(\mathbb{C})$

Thank you!