# Higher Recursion in Computable Structure Theory.

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# Summary

- **1**  $\Pi^1_1$ -ness and ordinals
- Output And A state of the st
- When hyperarithmetic is recursive
- Overspill
- **o** A structure equivalent to its own jump

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0, 1, 2, ...,

 $0,1,2,...,\,\omega,$ 

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 $0, 1, 2, ..., \omega, \omega + 1, \omega + 2, ...,$ 

 $0,1,2,...,\,\omega,\,\omega+1,\omega+2,...,\omega+\omega$ 

 $0, 1, 2, ..., \omega, \omega + 1, \omega + 2, ..., \omega + \omega = \omega 2,$ 

 $0,1,2,...,\,\omega,\,\omega+1,\omega+2,...,\omega+\omega=\omega 2,\omega 2+1,$ 

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0, 1, 2, ...,  $\omega$ ,  $\omega$  + 1,  $\omega$  + 2, ...,  $\omega$  +  $\omega$  =  $\omega$ 2,  $\omega$ 2 + 1,  $\omega$ 2 + 2 ...,  $\omega$ 3, ...,  $\omega$ 4, ...,

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#### Definition:

A linear ordering  $(A; \leq_A)$  is *well-ordered* if every subset has a least element.

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- A is isomorphic to an initial segment of B
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The class of ordinals is itself well-ordered by embeddability.

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The key notion connecting  $\Pi_1^1$ -ness and well-orders

is well-founded trees.

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$$\mathsf{rk}(T) = \sup_{n \in \mathbb{N}} (\mathsf{rk}(T_n) + 1),$$

where  $T_n = \{ \sigma \in \mathbb{N}^{<\omega} : n^{\frown} \sigma \in T \}.$ 

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The rank function is NOT a computable function.

Lemma: Given trees S and T,  $\mathsf{rk}(S) \leq \mathsf{rk}(T) \iff$  there is an  $\subsetneq$ -preserving embedding  $S \to T$ .

Definition: Given a linear ordering  $\mathcal{L} = (L; \leq_L)$ , define the *tree of descending sequences*:

$$T_{\mathcal{L}} = \{ \langle \ell_0, ..., \ell_k \rangle \in L^{<\omega} : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k \}.$$

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Furthermore, if  $\mathcal{L}$  is well-ordered,  $\mathsf{rk}(\mathcal{T}_{\mathcal{L}}) \cong \mathcal{L}$ .

Corollary: Deciding if a liner ordering is WO, is as hard as deciding if a tree is WF.

The *Kleene-Brower* ordering on  $\mathbb{N}^{<\omega}$  is defined as follows:

$$\sigma \leq_{\mathsf{KB}} \tau \quad \iff \quad \sigma \supseteq \tau \quad \lor \quad \exists i \ (\sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i)).$$

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Lemma:  $\operatorname{rk}(T) + 1 \leq (T; \leq_{KB}) \leq \omega^{\operatorname{rk}(T)} + 1$ .

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#### **Proof:**

• Every  $\Sigma_1^1$  formula  $\varphi(n)$  is equivalent to  $\exists f \in \mathbb{N}^{\mathbb{N}} \ \theta(f, n)$  where  $\theta$  is  $\Pi_1^0$ .

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- Every Σ<sub>1</sub><sup>1</sup> formula φ(n) is equivalent to ∃f ∈ ℕ<sup>ℕ</sup> θ(f, n) where θ is Π<sub>1</sub><sup>0</sup>.
- For a  $\Pi_1^0$  formula  $\theta(f)$ , there is a computable tree T with  $\theta(f) \iff f \in [T]$ .

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Omega-one-Church-Kleene

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$$\omega_1^{CK} \cong \sum_{e \in \mathcal{O}} \mathcal{L}_e$$

where  $\mathcal{L}_e$  is the linear ordering with index e.

Theorem: Let  $A \subset \mathcal{O}$  be  $\Sigma_1^1$ . There is an ordinal  $\alpha < \omega_1^{CK}$  such that  $\mathcal{L}_e < \alpha$  for all  $e \in A$ .

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Theorem: Let  $\mathfrak{A} \subset 2^{\mathbb{N}}$  be a  $\Sigma_1^1$  set of well-orderings of  $\mathbb{N}$ . There is an ordinal  $\alpha < \omega_1^{\mathcal{CK}}$  such that all  $\mathcal{L} < \alpha$  for all  $\mathcal{L} \in \mathfrak{A}$ .

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### Part II

- **1**  $\Pi^1_1$ -ness and ordinals
- Output A state of the state
- When hyperarithmetic is recursive
- Overspill
- **o** A structure equivalent to its own jump

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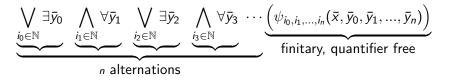
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We count alternations of  $\exists$  and  $\bigvee$  versus  $\forall$  and  $\bigwedge$ .

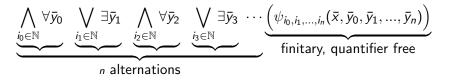
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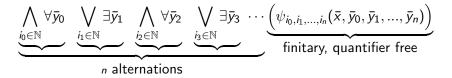
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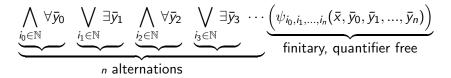
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We use  $\mathcal{L}_{c,\omega}$  to denote the set of computably infinitary formulas.

#### more examples

Example: There is a  $\Pi_{2\alpha+1}^{c}$  formula  $\psi_{\alpha}$  such that, on a partial ordering  $\mathcal{P}$ ,

$$\mathcal{P} \models \psi_{\alpha}(a) \quad \iff \quad \mathsf{rk}_{\mathcal{P}}(a) \leq \alpha.$$

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#### Hyperarithmetic sets

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Theorem: Let  $A \subseteq \mathbb{N}$ . The following are equivalent:

- A is definable by a  $\mathcal{L}_{c,\omega}$  formula
- There is a computable list  $\{\varphi_n : n \in \mathbb{N}\}$  of  $\mathcal{L}_{c,\omega}$  sentences

over the empty vocabulary  $\{\top, \bot\}$ 

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Theorem: [Kleene] Let  $A \subseteq \omega$ . The following are equivalent:

- A is hyperarithmetic
- A is Δ<sup>1</sup><sub>1</sub>.

• 
$$A \leq_m \mathcal{O}_{(\leq \alpha)}$$
 for some  $\alpha < \omega_1^{CK}$ 

## Transfinite iterations of the Turing jump

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Let  $\mathcal{L}$  be a well-ordering with domain  $\subseteq \mathbb{N}$ .

Definition: A *jump hierarchy* on  $\mathcal{L}$  is a set  $H \subseteq \mathcal{L} \times \mathbb{N}$  such that

$$H^{[\ell]} = (H^{[<\ell]})',$$

where  $X^{[\ell]} = \{x : (\ell, x) \in X\}$  and  $X^{[<\ell]} = \{(k, x) : k <_{\mathcal{L}} \ell \& (k, x) \in X\}.$ 

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Obs: For every well-ordering  $\mathcal{L}$  there is a unique jump hierarchy on it.

**Pf:** Show that there is an isomorphism  $\alpha \to \beta$  computable in both  $H_{\alpha}$  and  $H_{\beta}$ .

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For  $\alpha \in \omega_1^{CK} \smallsetminus \mathbb{N}$ :  $\mathcal{O}_{(<\omega^\alpha)} \equiv_{\mathcal{T}} 0^{(2\alpha+1)}$ .

# Part III

- **1**  $\Pi^1_1$ -ness and ordinals
- Output And A state of the st
- When hyperarithmetic is recursive
- Overspill
- **o** A structure equivalent to its own jump

Theorem: [Spector 55] If  $\leq_A \subseteq \omega^2$  is a hyperarithmetic well-ordering of  $\omega$ , then  $\mathcal{A} = (\omega; \leq_A)$  is isomorphic to a computable well-ordering.

**Proof:** 

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Theorem: If an infinitary formula has a hyperarithmetic representation it is equivalent to a computable infinitary formula

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Hyperarithmetic groups have UIm rank  $\leq \omega_1^{CK}$ . If the UIm rank is  $< \omega_1^{CK}$  use the computable operator. If the UIm rank is  $\omega_1^{CK}$ , we need to show their divisible part must be isomorphic to  $\mathbb{Q}^{\infty}$ , and hence they are bi-embeddable with  $\mathbb{Q}^{\infty}$ .

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By "relative to every oracle on a cone"

we mean "  $(\exists Y \in 2^{\omega})(\forall X \geq_T Y)$  the following holds relativized to Y."

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Def: E satisfies hyperarithmetic-is-recursive trivially

if every real is *E*-equivalent to a computable one.

#### The question

Question: What makes an equivalence relation

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**Question:** What makes an equivalence relation satisfy hyperarithmetic-is-recursive on a cone?

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**Obs:** Since in computability theory most proofs relativize: For "natural" E, E satisfies hyperarithmetic-is-recursive  $\iff$  it does on a cone.

## A sufficient condition: a first attempt

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A sufficient condition for hyp-is-rec.

**Def:** For  $\mathfrak{K} \subseteq 2^{\omega}$ ,  $(\mathfrak{K}, \equiv, r)$  is a ranked equivalence relation if  $\equiv$  is an equivalence relation on  $\mathfrak{K}$ , and  $r: \mathfrak{K}/\equiv \to \omega_1$ .

**Def:**  $(\mathfrak{K}, \equiv, r)$  is *scattered* if  $r^{-1}(\alpha)$  contains countably many equivalence classes for each  $\alpha \in \omega_1$ .

**Def:**  $(\mathfrak{K}, \equiv, r)$  is *projective* if  $\mathfrak{K}$  and  $\equiv$  are projective and r has a projective presentation  $2^{\omega} \rightarrow 2^{\omega}$ .

#### Theorem ([M.] (ZFC+PD))

Let  $(\mathfrak{K}, \equiv, r)$  be scattered projective ranked equivalence relation

such that  $\forall Z \in \mathfrak{K}, \ r(Z) < \omega_1^Z$ .

For every X on a cone, (i.e.  $\exists Y \forall X \ge_T Y$ ,) every equivalence class with an X-hyperarithmetic member has an X-computable member.

**Lemma:** [Martin] (ZFC+PD) If  $f: 2^{\omega} \to \omega_1$  is projective and  $f(X) < \omega_1^X$ , then f is constant on a cone.

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#### The main theorem

Theorem ([M. 13] (ZFC +  $(0^{\sharp} \text{ exists}) + \neg CH$ ) Let *E* be a  $\Sigma_1^1$ -equivalence relation on  $2^{\omega}$ . TFAE

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  - **1** E satisfies hyperarithmetic-is-recursive on a cone non-trivially.

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This theorem applies to all the examples mentioned before. Examples:

- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught's conjecture;
- $X \equiv Y \iff \omega_1^X = \omega_1^Y$ .

## The $\neg$ CH assumption.

Recall: E has perfectly many classes if there is a perfect tree all whose paths are E-inequivalent.

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Theorem ([M. 13] (ZFC + (0^{\sharp} \text{ exists}))
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Let E be a  $\Sigma_1^1$ -equivalence relation on  $2^{\omega}$ . TFAE

- E satisfies hyperarithmetic-is-recursive on a cone.
- 2 E does not have perfectly many equivalence classes.

#### The sharp assumption

**Def:**  $S \subseteq 2^{\omega}$  is cofinal (in the Turing degrees) if  $\forall Y \exists X \ge_T Y \ (X \in S)$ .

**Thm:** [Martin]( $0^{\ddagger}$  exists). If S is degree invariant and cofinal, it contains a cone.

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Theorem ([M. 13] (ZF)) Let  $\mathcal{E}$  be a  $\Sigma_1^1$ -equivalence relation on  $2^{\omega}$ . TFAE

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Theorem ([M. 13] (ZF))

Let  $\mathcal{E}$  be a  $\Sigma_1^1$ -equivalence relation on  $2^{\omega}$ . TFAE

E satisfies hyperarithmetic-is-recursive relative to a cofinal set of oracles.

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#### Theorem ([M. 13])

The following are equivalent over ZF.

- Every Σ<sub>1</sub><sup>1</sup>-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.
- $\bigcirc$  0<sup> $\ddagger$ </sup> exists.

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The key result in this proof is: Thm: [Sami 99] Let  $S = \{Y \in 2^{\omega} : \exists Z \ (\forall W \leq_{hyp} Z \ (W \leq_T Y) \& \omega_1^Z = \omega_1^Y\}.$ If S contains a cone, then  $0^{\sharp}$  exists.

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**The proof of our result** uses the following equivalence:  $X \equiv Y$  iff

- X and Y are code structures L<sub>α</sub>(A) and L<sub>β</sub>(B) with α = β and ω<sub>1</sub><sup>A</sup> = ω<sub>1</sub><sup>B</sup>,
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#### Theorem ([M. 13])

The following are equivalent over ZF.

 Every Σ<sub>1</sub><sup>1</sup>-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.

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It then uses Barwise compactness to put us in the hypothesis of Sami's theorem:

## Part IV

- **1**  $\Pi^1_1$ -ness and ordinals
- Output And A state of the st
- When hyperarithmetic is recursive
- Overspill
- **o** A structure equivalent to its own jump

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## Ill-founded hierarchies

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It has a computable presentation, but the  $\omega_1^{CK}$  cut is not even hyp.

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Corollary: [Kreisel] Let S be a  $\Pi_1^1$  set of  $\mathcal{L}_{c,\omega}$ -formulas. If every hyperarithmetic subset of S is satisfiable, then so is S.

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Given a structure  $\mathcal{A}$ , we define  $\mathcal{A}'$  by adding relations  $R_{i,j}$  for  $i, j \in \omega$ ,

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Examples:

- If  $\mathcal{L}$  a Linear ordering, then  $\mathcal{L}' \equiv (\mathcal{L}, succ, 0')$ .
- If  $\mathcal{B}$  a Boolean algebra, then  $\mathcal{B}' \equiv (\mathcal{B}, atom, 0')$ .

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Question: Is there a structure equivalent to its own jump?

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#### Theorem ([Montalbán 2011])

Infinitely many iterates of the power set are needed to prove this theorem.

Def: A jump-hierarchy structure is a structure  $\mathcal{J} = (A; \leq, R_{i,j} : i, j \in \omega)$ , where  $\mathcal{A} = (A; \leq)$  is a linear ordering and

 $\mathcal{A} \models R_{i,j}(a, b_1, ..., b_j) \iff b_1, ..., b_j < a \quad \& \quad \mathcal{L} \upharpoonright a \models \varphi_{i,j}^{\Sigma}(b_1, ..., b_j).$ 

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Obs: If A is well-ordered, there is a jump-hierarchy structure over A.

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Claim 5: There is a model  $\mathcal{B}$  of 1-8 with  $\omega_1^{\mathcal{B}} = \omega_1^{\mathcal{CK}}$  and  $\mathcal{L} \cong \mathcal{H}$ .

**Proof:** Use Barwise compactness.

### The necessity of infinitely many iterations of the power set

Theorem: [Montalbán 11]  
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**Proof** of case  $n = 1$ : Show that if  $\mathcal{A} \equiv_{Muchnik} \mathcal{A}'$ , then  
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