Higher Recursion in Computable Structure Theory.

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Workshop on Higher Recursion Theory
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Summary

1. Π^1_1-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump

Part I

1. Π^1_1-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump

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The class of ordinals is itself well-ordered by embeddability.

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The key notion connecting Π^1_1-ness and well-orders

is well-founded trees.

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The rank function is NOT a computable function.

Lemma: Given trees S and T,
$\mathrm{rk}(S) \leq \mathrm{rk}(T) \iff$ there is an $\subsetneq$-preserving embedding $S \to T$.

From linear orderings to trees

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Definition: Given a linear ordering $\mathcal{L} = (L; \leq_L)$,
define the *tree of descending sequences*:

$$T_{\mathcal{L}} = \{\langle \ell_0, ..., \ell_k \rangle \in L^{<\omega} : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k\}.$$

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Corollary: Deciding if a liner ordering is WO,
is as hard as deciding if a tree is WF.

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The *Kleene-Brower* ordering on $\mathbb{N}^{<\omega}$ is defined as follows:

$$\sigma \leq_{KB} \tau \quad \iff \quad \sigma \supseteq \tau \quad \vee \quad \exists i \; \big(\sigma \restriction i = \tau \restriction i \wedge \sigma(i) < \tau(i)\big).$$

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Lemma: $\mathrm{rk}(T) + 1 \leq (T; \leq_{KB}) \leq \omega^{\mathrm{rk}(T)} + 1$.

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$$\omega_1^{CK} \cong \sum_{e \in \mathcal{O}} \mathcal{L}_e$$

where $\mathcal{L}_e$ is the linear ordering with index e.

Σ^1_1-bounding

Theorem: Let $A \subset \mathcal{O}$ be Σ^1_1.
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- Define $\alpha = \sum_{n\in\mathbb{N}}(T_{\mathcal{L}_e} * S_n; \leq_{KB})$.

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Part II

1. Π^1_1-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump

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Theorem: [Scott 65] For every automorphism invariant set $B \subset \mathcal{A}^k$, there is an infinitary formula $\varphi(\bar{x})$ such that $B = \{\bar{b} \in A^k : \mathcal{A} \models \varphi(\bar{b})\}$.

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We use $\mathcal{L}_{c,\omega}$ to denote the set of computably infinitary formulas.

more examples

Example: There is a $\Pi^c_{2\alpha+1}$ formula ψ_α such that, on a partial ordering $\mathcal{P}$,

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$$\mathcal{L} \models \varphi_{\omega^\alpha} \iff \mathcal{L} \leq \omega^\alpha.$$

Hyperarithmetic sets

Definition: A set $A \subseteq \mathbb{N}$ is *hyperarithmetic* if
it is definable by an infinitary computable formula $\varphi(x)$.

$$A = \{n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) \models \varphi(n)\}.$$

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Theorem: Let $A \subseteq \mathbb{N}$. The following are equivalent:

- A is definable by a $\mathcal{L}_{c,\omega}$ formula
- There is a computable list $\{\varphi_n : n \in \mathbb{N}\}$ of $\mathcal{L}_{c,\omega}$ sentences over the empty vocabulary $\{\top, \bot\}$ such that $A = \{n \in \mathbb{N} : \models \varphi_n\}$.

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Theorem: [Kleene] Let $A \subseteq \omega$. The following are equivalent:

- A is hyperarithmetic
- A is Δ^1_1.
- $A \leq_m \mathcal{O}_{(\leq\alpha)}$ for some $\alpha < \omega_1^{CK}$

Transfinite iterations of the Turing jump

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Let $\mathcal{L}$ be a well-ordering with domain $\subseteq \mathbb{N}$.

Definition: A *jump hierarchy* on $\mathcal{L}$ is a set $H \subseteq \mathcal{L} \times \mathbb{N}$ such that

$$H^{[\ell]} = (H^{[<\ell]})',$$

where $X^{[\ell]} = \{x : (\ell, x) \in X\}$ and $X^{[<\ell]} = \{(k, x) : k <_{\mathcal{L}} \ell \ \& \ (k, x) \in X\}$.

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Obs: For every well-ordering $\mathcal{L}$ there is a unique jump hierarchy on it.

Different presentations

Theorem: Suppose α and β are different presentations of the same ordinal.
Let H_α and H_β be the jump hierarchies on them.
Then $H_\alpha \equiv_T H_\beta$.

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For $\alpha \in \omega_1^{CK} \smallsetminus \mathbb{N}$: $\mathcal{O}_{(<\omega^\alpha)} \equiv_T 0^{(2\alpha+1)}$.

Part III

1. Π^1_1-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump

Every hyperarithmetic well-ordering is computable

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Theorem: [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of ω, then $\mathcal{A} = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

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Theorem: If an infinitary formula has a hyperarithmetic representation
it is equivalent to a computable infinitary formula

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We produce bi-embeddability invariants for linear orderings given by finite trees with ordinal labels. Finally, we build a computable map from invariants to linear orderings.

Another similar behavior

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Hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$. If the Ulm rank is $< \omega_1^{CK}$ use the computable operator. If the Ulm rank is ω_1^{CK}, we need to show their divisible part must be isomorphic to $\mathbb{Q}^\infty$, and hence they are bi-embeddable with $\mathbb{Q}^\infty$.

Counterexample to Vaught's conjecture

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Let T be an $L_{\omega_1,\omega}$ sentence with uncountably many models. TFAE

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By "relative to every oracle on a cone"
we mean " $(\exists Y \in 2^\omega)(\forall X \geq_T Y)$ the following holds relativized to Y."

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An equivalence class E on 2^ω satisfies *hyperarithmetic-is-recursive* if every hyperarithmetic real is E-equivalent to a computable one.

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Def: E satisfies hyperarithmetic-is-recursive *trivially* if every real is E-equivalent to a computable one.

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If E is Σ^1_1 and we define $X\ F\ Y \iff (X\ E\ Y) \vee (\omega_1^X = \omega_1^Y = \omega_1^{CK})$,
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Question: What makes an equivalence relation
satisfy hyperarithmetic-is-recursive on a cone?

Martin's measure

Def: A *cone* is a set of the form $\{X \in 2^{\mathbb{N}} : X \geq_T Y\}$ for some $Y \in 2^{\mathbb{N}}$.

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Def: E satisfies *hyperarithmetic-is-recursive on a cone* if,
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Obs: Since in computability theory most proofs relativize:
For "natural" E,
E satisfies hyperarithmetic-is-recursive $\iff$ it does on a cone.

A sufficient condition: a first attempt

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A sufficient condition for hyp-is-rec.

Def: For $\mathfrak{K} \subseteq 2^\omega$, $(\mathfrak{K}, \equiv, r)$ is a *ranked equivalence relation* if
$\equiv$ is an equivalence relation on $\mathfrak{K}$, and $r \colon \mathfrak{K}/\equiv \to \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *scattered* if
$r^{-1}(\alpha)$ contains countably many equivalence classes for each $\alpha \in \omega_1$.

Def: $(\mathfrak{K}, \equiv, r)$ is *projective* if
$\mathfrak{K}$ and $\equiv$ are projective and r has a projective presentation $2^\omega \to 2^\omega$.

Theorem ([M.] (ZFC+PD))

Let $(\mathfrak{K}, \equiv, r)$ be scattered projective ranked equivalence relation such that $\forall Z \in \mathfrak{K},\ r(Z) < \omega_1^Z$.
For every X on a cone, (i.e. $\exists Y \forall X \geq_T Y$,) every equivalence class with an X-hyperarithmetic member has an X-computable member.

Lemma: [Martin] (ZFC+PD) If $f \colon 2^\omega \to \omega_1$ is projective and $f(X) < \omega_1^X$, then f is constant on a cone.

The main theorem

Theorem ([M. 13] (ZFC + ($0^\sharp$ exists) + $\neg$CH)

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This theorem applies to all the examples mentioned before.

Examples:

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The ¬CH assumption.

The $\neg$CH assumption.

Theorem: [Burgess 78] Let E be Σ^1_1-equivalence relation on 2^ω.
Either E has perfectly many classes, **or** it has at most $\aleph_1$ many classes.

Recall: E has *perfectly many classes* if there is a perfect tree all whose paths are E-inequivalent.

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1. *E satisfies hyperarithmetic-is-recursive on a cone.*
2. *E does not have perfectly many equivalence classes.*

The sharp assumption

Def: $\mathcal{S} \subseteq 2^\omega$ is *cofinal (in the Turing degrees)* if $\forall Y\ \exists X \geq_T Y\ (X \in \mathcal{S})$.

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Theorem ([M. 13] (ZF))

Let $\mathcal{E}$ be a Σ^1_1-equivalence relation on 2^ω. TFAE

1. *E satisfies hyperarithmetic-is-recursive relative to a cofinal set of oracles.*

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Thm: [Martin]($0^\sharp$ exists). If $\mathcal{S}$ is degree invariant and cofinal, it contains a cone.

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Let $\mathcal{E}$ be a Σ^1_1-equivalence relation on 2^ω. TFAE

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The proof of our result uses the following equivalence: $X \equiv Y$ iff

- X and Y are code structures $L_\alpha(A)$ and $L_\beta(B)$ with $\alpha = \beta$ and $\omega_1^A = \omega_1^B$,
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It then uses Barwise compactness to put us in the hypothesis of Sami's theorem:

Part IV

1. Π^1_1-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump

Ill-founded hierarchies

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It has a computable presentation, but the ω_1^{CK} cut is not even hyp.

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Theorem: [Barwise] Let $\{\varphi_{f(e)} : e\in\omega_1^{CK}\}$ be a computable list of $\mathcal{L}_{\mathsf{c},\omega}$-sentences such that, for every $\alpha<\omega_1^{CK}$, $\{\varphi_{f(e)} : e<\alpha\}$ is satisfiable.

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Corollary: [Kreisel] Let S be a Π_1^1 set of $\mathcal{L}_{c,\omega}$-formulas.
If every hyperarithmetic subset of S is satisfiable, then so is S.

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Proof: The set of countable models of ZFC & $\forall x \in \omega\ (\bigvee\!\!\!\bigvee_{n \in \mathbb{N}} x = \mathbf{n})$ is Σ_1^1.
So there is such a model with $\omega_1^{\mathcal{M}} = \omega_1^{CK}$.

Part V

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2. Hyperarithmeticy
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5. A structure equivalent to its own jump

The jump of a structure

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Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, ..., a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi^{\Sigma}_{e,j}(a_1, ..., a_j),$$

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Examples:

- If $\mathcal{L}$ a Linear ordering, then $\mathcal{L}' \equiv (\mathcal{L}, succ, 0')$.
- If $\mathcal{B}$ a Boolean algebra, then $\mathcal{B}' \equiv (\mathcal{B}, atom, 0')$.

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Uses the existence of $0^\sharp$ and builds a model of $ZFC + V = L$.

Proof 2: [Puzarenko 2011]
Uses ω_1^{CK} iterates of power set and builds a model of $KP + V = L$.

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Theorem ([Montalbán 2011])
Infinitely many iterates of the power set are needed to prove this theorem.

Jump-hierarchy structures

Def: A jump-hierarchy structure is a structure $\mathcal{J} = (A; \leq, R_{i,j} : i, j \in \omega)$, where $\mathcal{A} = (A; \leq)$ is a linear ordering and

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Obs: If $\mathcal{A}$ is well-ordered, there is a jump-hierarchy structure over $\mathcal{A}$.

The Hanf number of computably infinitary logic

Lemma: [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence φ has a model of size $\beth_{\omega_1^{CK}}$,
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Claim 5: There is a model $\mathcal{B}$ of 1-8 with $\omega_1^{\mathcal{B}} = \omega_1^{CK}$ and $\mathcal{L} \cong \mathcal{H}$.

Proof: Use Barwise compactness.

The necessity of infinitely many iterations of the power set

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Proof of case $n = 1$: Show that if $\mathcal{A} \equiv_{Muchnik} \mathcal{A}'$, then

$$\{X \subseteq \omega : X \text{ is c.e. in every copy of } \mathcal{A}\}$$

forms an ω-model of 2nd-order arithmetic.