Homeomorphisms of Cantor space

and automorphisms of the Turing degrees

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Complementation

$A \mapsto \omega \setminus A$ induces a nontrivial automorphism of D_m. It is also a homeomorphism that uniformly induces a map on $2^\omega / =^*$. This implies that it must be computable, but not that it is trivial as an automorphism of D_m.

The Turing degrees

- Slaman and Woodin results on automorphisms:
  - At most countably many
  - $\pi(\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x} \geq \mathbf{0}''$

Inducing automorphisms

If a nontrivial $\pi \in \mathrm{Aut}(\mathcal{D})$ exists:

1. Can it be induced by $F : 2^\omega \to 2^\omega$ (a continuous map)?
2. Can it be induced by $F : 2^{<\omega} \to 2^{<\omega}$ (a tree isomorphism)
3. Can it be induced by $f : \omega \to \omega$ (a permutation)?
4. Can it be induced by a polynomial?
5. Can it be induced by an analytic function $f : [0,1] \to [0,1]$?

Answers

1 Not known

2 Not known (may be easy)

3 No (K. 2018)

4 No (easy)

5 Not known to me

The Turing degrees

Definition

For $A, B \subseteq \omega$, we say that $A \leq_T B$ if A can be computed from B. If $A \leq_T B$ and $B \leq_T A$ then we say that $A \equiv_T B$. If $\mathbf{a} = \{X \mid X \equiv_T A\}$ denotes the Turing degree of A and we define $\mathbf{a} \leq \mathbf{b} \leftrightarrow A \leq_T B$, then $\mathcal{D} = \{\mathbf{a} \mid A \subseteq \omega\}$ is the upper semilattice of Turing degrees.

Automorphisms, subalgebras, ideals

Note that 0 and $\leq$ are definable from the l.u.b. operation $\vee$. Shore and Slaman (1999) showed that so is the map $\mathbf{a} \mapsto \mathbf{a}'$. Here are a few natural things we should know about the algebra $\langle \mathcal{D}, \vee \rangle$, in order to say that we understand it.

Automorphisms - bijections $\pi : \mathcal{D} \to \mathcal{D}$ such that $\pi(\mathbf{x} \vee \mathbf{y}) = \pi(\mathbf{x}) \vee \pi(\mathbf{y})$.

Subalgebras - subsets of $\mathcal{D}$ closed under $\vee$.

Ideals - sets $\mathcal{I} \subseteq \mathcal{D}$ that are "ideal elements" in the sense that if we write $\mathbf{a} \leq \mathcal{I}$ for $\mathbf{a} \in \mathcal{I}$, then $\mathbf{a} \leq \mathbf{b} \leq \mathcal{I} \Rightarrow \mathbf{a} \leq I$, and $\mathbf{a} \leq \mathcal{I}$, $\mathbf{b} \leq \mathcal{I} \Rightarrow \mathbf{a} \vee \mathbf{b} \leq \mathcal{I}$.

Groszek and Slaman (1983) showed that isomorphism types of subalgebras and ideals are not determined by ZFC. Open problem: Does $\langle \mathcal{D}, \vee \rangle$ have any nontrivial automorphism?

Bernoulli measures

For each $n \in \omega$,

$$\mu_p(\{X : X(n) = 1\}) = p$$

$$\mu_p(\{X : X(n) = 0\}) = 1 - p$$

and $X(0), X(1), X(2), \ldots$ are mutually independent random variables.

The Turing degrees

Two approaches:

- $\mathcal{D}_T = \omega^\omega / \equiv_T$
- $\mathcal{D}_T = 2^\omega / \equiv_T$

Same abstract structure, different notions of "inducing".

Open problem

Question

Does $(\mathcal{D}_T, \leq)$ have any nontrivial automorphisms?

- $\pi : \mathcal{D}_T \to \mathcal{D}_T$ is an automorphism if it is bijective and $\mathbf{x} \leq \mathbf{y} \iff \pi(\mathbf{x}) \leq \pi(\mathbf{y})$.
- $\pi : \mathcal{D}_T \to \mathcal{D}_T$ is *nontrivial* if $(\exists \mathbf{x})(\pi(\mathbf{x}) \neq \mathbf{x})$.

Other degree structures

- The hyperdegrees $\mathcal{D}_h$ have no nontrivial automorphisms (Slaman, Woodin $\sim$1990).
- The Turing degrees $\mathcal{D}_T$ have at most countably many (Slaman, Woodin $\sim$1990).
- The many-one degrees $\mathcal{D}_m$ have many automorphisms.

History of Aut(D_T).

1980 Nerode and Shore show each automorphism equals the identity on some cone.

1990 Slaman and Woodin announce and circulate proofs that the cone can be lowered to $0''$, and Aut(D_T) is countable.

1999 Cooper sketches a construction of a nontrivial automorphism, but does not finish that project. Proposed automorphism π is induced by a continuous map on ω^ω.

2008 Outline of Slaman-Woodin results published.

2018 No automorphisms induced by permutations.

Inducing, from ω to 2^ω to $\mathcal{D}_T$

Definition

The pullback of $f : \omega \to \omega$ is $f^* : \omega^\omega \to \omega^\omega$ given by

$$f^*(A)(n) = A(f(n)).$$

We often write $F = f^*$.

$$\pi([A]_T) = [F(A)]_T$$

Plausible that a permutation would induce an automorphism?

Theorem (Haught and Slaman 1993)

A permutation of ω (actually $2^{<\omega}$) can induce an automorphism of

$$(PTIME^A, \leq_{pT}).$$

Caveat: the automorphism is probably not in the ideal itself.

Plausible that a permutation would induce an automorphism?

Theorem (Kent ∼1967)

There exists a permutation f such that

(i) *for all recursively enumerable B, $f(B)$ and $f^{-1}(B)$ are recursively enumerable (and hence for all recursive A, $f(A)$ and $f^{-1}(A)$ are recursive);*

(ii) *f is not recursive.*

So a noncomputable f may map the Turing degree $\mathbf{0}$ to $\mathbf{0}$.

Definition

$A \subset \omega$ is cohesive if for each recursively enumerable set W_e, either $A \cap W_e$ is finite or $A \cap (\omega \setminus W_e)$ is finite.

Proof.

Kent's permutation is just any permutation of a cohesive set (and the identity off the cohesive set). □

The case $D_T = \omega^\omega / \equiv_T$ is trivial

$$f^*(f^{-1})(n) = f^{-1}(f(n)) = n$$

so

$$f^{-1} \mapsto_{f^*} \mathrm{id}_\omega$$

$\therefore$ f^* maps f^{-1} to a computable function $\therefore$ f^{-1} is computable $\therefore$ f is computable

For 2^ω, one idea is: think of the elements of 2^ω as probabilities.

Theorem (K. 2010)

Each μ_p-random computes p (layerwise!).

The idea now is that the permutation f of ω preserves something, namely μ_p for any p.

Theorem

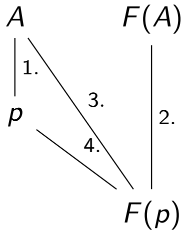
A permutation $f : \omega \to \omega$ induces an automorphism of $\mathcal{D}_T$ iff f is computable.

Two proof steps.

First show f induces the trivial automorphism. Then use that to show f is computable. $\square$

Steps of the proof

Assume A is F-μ_p-ML-random.



1. $p \leq_T A$ (Law of the Iterated Logarithm)
2. $F(p) \leq_T F(A)$
3. $F(p) \leq_T A$
4. $F(p) \leq_T p$ (Lebesgue Density Theorem & Sacks/de Leeuw, Moore, Shannon, Shapiro)

Majority vote computation of F

If F induces the trivial automorphism of $\mathcal{D}_T$, we prove F is computable.
Notation: $A + n = A \cup \{n\}$, $A - n = A \setminus \{n\}$.
We use Lebesgue Density again, this time for $p = 1/2$.

We have $F(A) \leq_T A$. Fix Φ which works for $1 - \frac{\varepsilon}{2}$ measure many A.

$$\begin{array}{ccc}
F(A+n) & \overset{\mathbb{P}\geq 1-\varepsilon}{=\!=\!=\!=} & \Phi^{A+n} \\
\mathbb{P}=1 \Big| & & \Big| \therefore \mathbb{P}\geq 1-2\varepsilon \\
F(A-n) & \overset{\mathbb{P}\geq 1-\varepsilon}{=\!=\!=\!=} & \Phi^{A-n}
\end{array}$$

- $=$ means equal
- $-$ means a Hamming distance of 1.

New results (2019)

No nontrivial automorphism of D_r is represented by a permutation for any D_r between $\leq_1$ and $\leq_T$.

Homeomorphisms

The following kinds of homeomorphisms are excluded.

- Those that preserve all Bernoulli measures
- Those that uniformly induce maps on $2^\omega / =^*$

Here uniform means: for each a there is a b such that for all A, B, if A equals B on $[b, \infty]$ then $F(A)$ equals $F(B)$ on $[a, \infty]$.