Algebraic properties of elementary embeddings

Scott Cramer

California State University, San Bernardino

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$\rightarrow$: Do properties of left-distributive algebras imply interesting structural properties of large cardinals or have interesting set-theoretic applications?

Question: Are properties of left-distributive algebra useful for proving extensions of Woodin's AD-conjecture?

left-distributive algebras

Definition

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Theorem (Laver)

Suppose $j : V_\lambda \to V_\lambda$ is an elementary embedding. Then $\mathcal{A}_j \approx \mathcal{A}$, the free left-distributive algebra on one generator.

Laver's theorem establishes a connection between left-distributive algebras and large cardinals.

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Using this connection, one can now obtain properties of the free left-distributive algebra one one generator from large cardinals.

Theorem (Laver, Steel)

Suppose that there is an embedding $j : V_\lambda \to V_\lambda$. Then the ordering $<_L$, the transitivization of the left-divisor relation on $\mathcal{A}$, is a linear order.

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The large cardinal assumption was later removed by Dehornoy.

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For $j : V_\lambda \to V_\lambda$ an elementary embedding. The set

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This fact about critical points of elementary embeddings can be coded into a fact about finite left-distributive algebras.

Laver tables

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
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Laver showed from the existence of an elementary embedding $V_\lambda \to V_\lambda$ that the period of the 1st row of the Laver table of A_n goes to infinity as $n \to \omega$. This is a translation of Laver's theorem on the critical points in $\mathcal{A}_j$.

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Dougherty-Jech showed that this cannot be shown in primitive recursive arithmetic.

One way that has been suggested for extending Laver's theorem is answering the following question:

Question (Laver)

Assuming any large cardinal, do there exist $j, k : V_\lambda \to V_\lambda$ *such that the algebra* $\mathcal{A}_{j,k}$ *generated by* j *and* k *is the free left-distributive algebra on* 2 *generators?*

extending Laver's bridge: adding more generators

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Conjecture (Brooke-Tayler, C., Miller)

Assume there is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ and write j_n for the nth iterate of j. Then for k and ℓ defined by

$$k = j \circ j_1 \circ j_2 \circ \cdots , \qquad \ell = j \circ j \circ j_1 \circ j_2 \circ \cdots$$

we have $\mathcal{A}_{j,k}$ is the free left-distributive algebra on 2 generators.

We take the following approach to extending Laver's theorem:

Question

Given a collection of (arbitrarily strong) elementary embeddings E can E be embedded into a 'minimal' simply-definable left-distributive algebra A, preserving all of its algebraic structure?

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Example

Suppose that $j, k, \ell : V_\lambda \to V_\lambda$ are such that

$$j = k[k] = \ell[\ell] \text{ and } \ell \in \operatorname{rng}(k).$$

Can we embed $\mathcal{A}_{k,\ell}$ into a simply definable left-distributive algebra A?

Lemma

Suppose that $k_1, \ldots, k_n : V_\lambda \to V_\lambda$ *are such that* $k_1 k_1 = \cdots = k_n k_n = j$ *and for all* $1 < i \le n$,

$$k_1, \ldots, k_{i-1} \in rng\, k_i.$$

Then $\mathcal{A}_{k_1,\ldots,k_n}$ *embeds into* $\mathcal{A}$ *(i.e. there is an injective algebra homomorphism).*

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Define $\mathcal{A}_\omega$ to be the direct limit of the following system, where every embedding is $\pi : \mathcal{A} \to \mathcal{A}$, the embedding generated by $\pi(x) = xx$, for x the generator:

$$\mathcal{A} \to \mathcal{A} \to \mathcal{A} \to \cdots .$$

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Theorem

(C.) Suppose that $j^* : V_{\lambda+1} \to V_{\lambda+1}$ is an elementary embedding and $j, k_1, k_2, \dots : V_\lambda \to V_\lambda$ are such that $k_1 k_1 = k_2 k_2 = \cdots = j$ and for all $1 < i < \omega$,

$$k_1, \dots, k_{i-1} \in rng\, k_i.$$

Then $\mathcal{A}_{k_1,k_2,\dots}$ embeds into $\mathcal{A}_\omega$.

Now that we have a new connection between elementary embeddings and left-distributive algebras, can we prove any algebraic results from large cardinals or vice versa?

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Theorem (C.)

Assume there is an elementary embedding $L_\omega(V_{\lambda+1}) \to L_\omega(V_{\lambda+1})$. Suppose that $a \in \mathcal{A}_\omega$. Then the map $\pi_a : (\mathcal{A}_\omega, \cdot) \to (\mathcal{A}_\omega, \cdot)$ given by $\pi_a(b) = a \cdot b$ is elementary (but not surjective).

can we complete $\mathcal{A}_\omega$?

$\mathcal{A}_\omega$ has ordertype $\mathbb{Q}$.

Question

Is there a natural left-distributive algebra on the completion of $\mathcal{A}_\omega$?

the algebra $\mathcal{P}$

Laver showed that for $\mathcal{P}$, the free left-distributive algebra on one generator with composition $\circ$ satisfies the following.

Theorem (Laver)

For $p, q \in \mathcal{P}$, the element $p \circ q$ is the limit of the sequence

$$p, pq, pqp, pqp(pq), pqp(pq)(pqp), \ldots$$

An *inverse limit* in this context is a limit of the form

$$j_0 \circ j_1 \circ j_2 \circ \cdots .$$

Our theorem above showed that we can embed the embeddings of (certain) inverse limits into $\mathcal{A}_\omega$. But can we find the inverse limit in the completion of $\mathcal{A}_\omega$ and define its action appropriately?

what does $\mathcal{A}$ actually look like?

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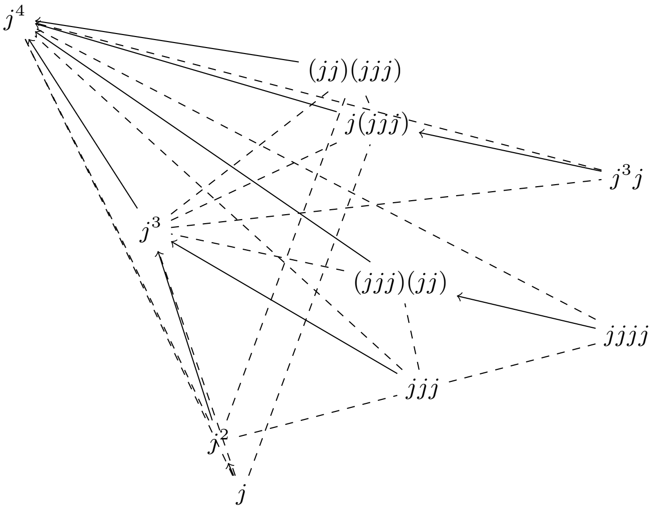
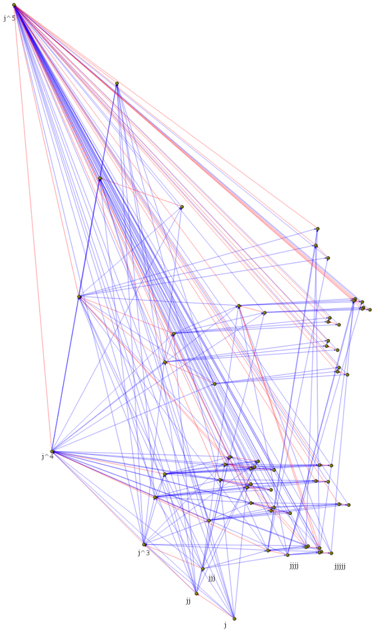
& j^2 \arrow[uul] \arrow[u, dash, dashed] & \\
& & j \arrow[ul]
\end{tikzcd}
$$

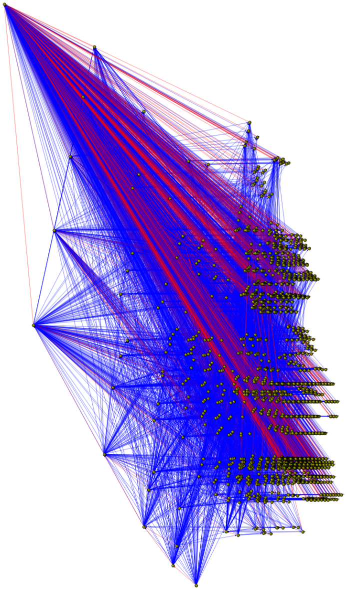
rank ≤ 4 elements in $\mathcal{A}_j$



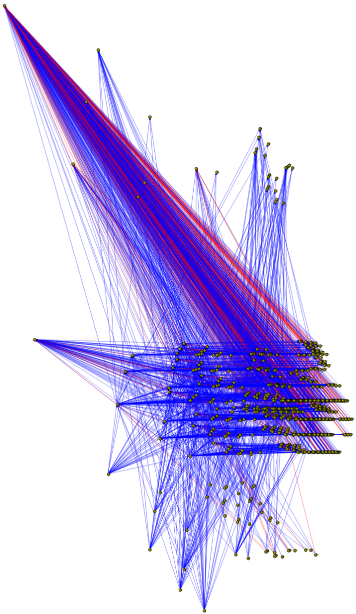
rank ≤ 5



rank ≤ 6



rank $\leq$ 6 gap



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Conjecture

For any $a \leq_L c <_L b \in \mathcal{A}$, there is no d with $b <_L d \leq ba$ such that c is a left-divisor of d.

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Conjecture

For any $a \leq_L c <_L b \in \mathcal{A}$, there is no d with $b <_L d \leq ba$ such that c is a left-divisor of d.

The case 'c divides b' is true.

Proposition (C.)

Assume there is an elementary embedding $L_\omega(V_{\lambda+1}) \to L_\omega(V_{\lambda+1})$. Then for any $a \leq_L c <_L b \in \mathcal{A}$ such that c left-divides b, there is no d with $b <_L d \leq ba$ such that c is a left-divisor of d.

The proof uses the elementarity of $\pi_c : \mathcal{A}_\omega \to \mathcal{A}_\omega$.

an extension of $\mathcal{A}_\omega$

Define $\mathcal{A}_\omega^*$ to be the completion of $\mathcal{A}_\omega$ under the $<_L$-topology. Extend application to $\mathcal{A}_\omega^*$ (as generated) by the definition

$$a{\cdot}b = \sup\{c \in \mathcal{A}_\omega \,|\, \forall d \in [b, a)(\text{`}d \text{ below } b\text{'} \rightarrow \forall e \in (a, c)\,(d \text{ does not left-divide } e))\}.$$

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‘d below b’ would be possible by ‘reading the gaps’.

Question

Does the extension $\mathcal{A}_\omega^*$ define application consistent with application in $\mathcal{A}_\omega$, and consistent with $\mathcal{P}_\omega$ (the result of adding composition to $\mathcal{A}_\omega$)?

Clearly, $\mathcal{A}_\omega^*$ will not succeed in capturing the entire algebra of embeddings (since $\mathcal{A}_\omega$ is countable).

extensions of $\mathcal{A}_\omega$

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But we can define many similar structures using direct limit systems. We skip the details of this and refer to these as $\mathcal{A}_R$ for various R.

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But we can define many similar structures using direct limit systems. We skip the details of this and refer to these as $\mathcal{A}_R$ for various R.

We can in fact define a maximal R, $R_{\max}$.

maximality of $A_{R_{\max}}$.

Lemma (C.)

If S is a finite set of embeddings which is rigid and square-closed then $\mathcal{A}_S$ embeds into $\mathcal{A}$.

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$\mathcal{A}_{R_{\max}}$ is linearly ordered by $<_L$.

questions about $\mathcal{A}_{R_{\max}}$

If the algebra does extend coherently to inverse limits (and other conjectures above are true), then you would expect to have an embedding

$$\mathcal{A}_2 \to \mathcal{A}_\omega^*.$$

obtaining properties of large cardinals from these algebras?

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Guess: Homogeneity properties of $\mathcal{A}_\omega^*$ might be useful for obtaining properties of inverse limits.