Wellordered Unions of Quotients of Smooth Equivalence Relations.

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This work is joint with William Chan.

We study wellordered (disjoint) unions of quotients of smooth equivalence relations in models of determinacy.

Definition

An equivalence relation E on a Polish space X is smooth if X/E injects into $\mathbb{R}$.

We say M is a natural model of determinacy if

$$M \models \mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R})).$$

- If M is a natural model of AD^+ then any set $A \in (V_\Theta)^M$ is the quotient of an equivalence relation on $\mathbb{R}$.
- Thus, a significant part of the theory of "definable sets" is captured by the theory of definable equivalence relations.

An equivalence relation E is said to be countable if every class $[x]_E$ is countable.

Remark

As in the Borel context, we have the Feldman-Moore theorem that every countable equivalence relation is induced by the action of a countable group.

The case of countable equivalence relations emcompasses all degree notions (e.g., Turing degrees, arithmetical degrees, L-degrees, $L[S]$-degrees) and so is of interest in descriptive set theory.

- A reasonable analog of "countable unions" in the Borel context is "wellordered union" in the general definable context.
- Another motivation comes from trying to show preservation of combinatorial properties under varions operations, including wellordered unions.

In particular, motivation comes from considering the Jónsson property.

Let $[X]^n$ denote the set of n-tuples of distinct elements from X.

Definition

X has the Jónsson property if for every $F\colon [X]^{<\omega} \to X$ there is a $Y \subseteq X$ with $Y \approx X$ (in bijection) such that $F[[Y]^{<\omega}] \neq X$.

- [J, Ketchersid, Schlutzenberg, Woodin]

  Assuming $\mathrm{AD} + V = L(\mathbb{R})$, every $\kappa < \Theta$ is Jónsson (Woodin extended this to models of AD^+).
- [Holshauser, J]

  Assuming AD, $\mathbb{R} \times \kappa$ is Jónsson for all $\kappa < \Theta$.

On the other hand, E_0 is 2-Jónsson [Holshausr, J], but not 3-Jónsson [Chan, Meehan].

This leads to the following question and conjecture:

Question

Assuming AD, which sets are Jónsson?

Conjecture

Assuming AD, every wellordered union of Jónsson sets is Jónsson.

Sets which are represented as quotients of smooth equivalence relations are the simplest, so it natural to ask:

Question

Does a wellordered union of quotients of smooth equivalence relations on $\mathbb{R}$ have the Jónsson property?

- There is also a connection between wellordered unions of quotients of smooth equivalence relations and work of Woodin on $\omega_1^{<\omega_1}$ which we mention below.
- We will see that there is a significant difference between the cases where E is countable or a general equivalence relation.

Our first main result is the following.

Theorem

Assume ZF + AD$^+$ + $V = L(\mathcal{P}(\mathbb{R}))$. *Let* $\kappa \in$ *On and* $\{E_\alpha\}_{\alpha<\kappa}$ *a sequence of smooth equivalence relations with all classes countable. Then* $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha \approx \mathbb{R} \times \kappa$.

Corollary

$\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ is Jónsson.

Remark

There will be two separate arguments to show $\mathbb{R} \times \kappa$ injects into $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ and to show $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ injects into $\mathbb{R} \times \kappa$.

Woodin analyzed the (uncountable) cardinals below ω_1^ω assuming $\mathsf{AD}_{\mathbb{R}}$. They are ω_1, $\mathbb{R}$, $\omega_1 \sqcup \mathbb{R}$, $\omega_1 \times \mathbb{R}$, and ω_1^ω.

- This result need $\mathsf{AD}_{\mathbb{R}}$. The cardinal structures below $\omega_1^{<\omega_1}$ under $\mathsf{AD} + V = L(\mathbb{R})$ and $\mathsf{AD}_{\mathbb{R}}$ are different.

Woodin also showed that the cardinal structure below $\omega_1^{<\omega_1}$ is extremely complicated.

Note that if $A = \sqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ where each E_α is smooth, then $\mathbb{R} \sqcup \omega_1$ injects into A.

The next result characterizes those $X \subseteq \omega_1^{<\omega_1}$ which can be represented as wellordered (disjoint) unions of quotients of smooth equivalence relations.

Theorem

Assume $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$*. Let* $X \subseteq \omega_1^{<\omega_1}$*. Then* $X \approx \sqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$*, where each* E_α *is a smooth equivalence relation on* $\mathbb{R}$*, iff* $\mathbb{R} \sqcup \omega_1$ *injects into* X*.*

Along the way we also show the following.

Theorem

Assume ZF + AD*. For any sequence* $\{E_\alpha\}_{\alpha<\kappa}$ *of countable equivalence relations on* $\mathbb{R}$*,* ω_1^ω *does not embed into* $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$*.*

Review of concepts

We use heavily aspects of Woodin's AD^+ theory.

For M a model of ZF and $S \subseteq \mathrm{On}$, $S \in M$, let ${}_n\mathbb{O}_S^M$ be the collection of subsets of $(\mathbb{R}^n)^M$ which are $(\mathrm{OD}_S)^M$.

Fact (Vopenka's theorem)

Let M be a transitive model of ZF and $S \in M$ a set of ordinals.

- *Every $x \in \mathbb{R}^M$ induces a $\mathbb{O}_S^M$ generic filter G_x over HOD_S^M. Also $x = \tau[G_x]$ for a canonical name τ.*
- *Let K be an OD_S^M set or ordinals and φ a formula. Let $p \in \mathbb{O}_S^M$ be $p = \{x \in \mathbb{R}^M : L[K, x] \models \varphi(K, x)\}$. Let N be a transitive inner model with $HOD_S^M \subseteq N$. Then $N \models p \Vdash_{\mathbb{O}_S^M} L[K, \tau] \models \varphi(K, \tau)$*

- If $(g_0, \ldots, g_{n-1})$ is ${}_n\mathbb{O}^M_S$-generic over N, then each g_i is $\mathbb{O}^M_S$-generic over N.

We have the following generalization of Silver's theorem due to Woodin.

Theorem (Woodin)

Assume $\mathrm{ZF} + \mathrm{AD}^+$. *Let* E *be an equivalence relation on* $\mathbb{R}$. *Then either*

1. $\mathbb{R}/E$ *is wellorderable*
2. $\mathbb{R}$ *injects into* $\mathbb{R}/E$.

$A \subseteq \mathbb{R}$ is ∞-Borel if there is an $S \subseteq \mathrm{On}$ and a formula φ such that for all $x \in \mathbb{R}$, $x \in A \leftrightarrow L[S, x] \models \varphi(S, x)$.

- Part of AD^+ asserts that every $A \subseteq \mathbb{R}$ is ∞-Borel.
- (Woodin) Assume $\mathrm{ZF} + \mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, and let $S \subseteq \mathrm{On}$. If $A \subseteq R$ is OD_S, than A has an OD_S ∞-Borel code.

If $A \subseteq \kappa \times \mathbb{R}$ we likewise say (S, φ) is an ∞-Borel code for A if $(\alpha, x) \in A \leftrightarrow L(S, x) \models \varphi(S, \alpha, x)$.

- Assuming $\mathrm{ZF} + \mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, every $A \subseteq \kappa \times \mathbb{R}$ has an ∞-Borel code.

Woodin also showed that a model of $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ is either a model of $\mathsf{AD}_{\mathbb{R}}$ or of the form $L(J, \mathbb{R})$ for some $J \subseteq \mathrm{On}$.

- In a model of $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ every set is OD from a set $S \subseteq \mathrm{On}$.

Lower Bound

Fact

Assume $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$*. Let* $\kappa \in$ *On and* $\{E_\alpha\}_{\alpha<\kappa}$ *a sequence of smooth equivalence relations on* $\mathbb{R}$ *with all classes countable. Then* $\mathbb{R} \times \kappa$ *injects into* $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$*.*

Remark

This lower-bound does not hold in general without the assumption that each E_α is a countable equivalence relation. A counterexample is given by

$$xE_\alpha y \leftrightarrow (x, y \notin \mathrm{WO}_\alpha) \vee (x = y).$$

Sketch of proof:
Let (S, φ) be an ∞-Borel code for $\{E_\alpha\}_{\alpha<\kappa}$. This uniformly gives ∞-Borel codes for the E_α.

For an eq. rel. E on $\mathbb{R}$ (with code S) there are 2 cases in the Woodin/Silver dichotomy:

Case 1. $\forall^* d \in \mathcal{D}\ \forall a \in L(S, d)$ there is some E-component $A \in \mathbb{O}_S^{L(S,d)}$ which contains a.

In this case we get a wellordering of $\mathbb{R}/E$ using the wellfoundedness of $\prod_{\mathcal{D}} \mathrm{On}/\nu$ (Martin measure on $\mathcal{D}$).

Case 2. There is an $x \in \mathbb{R}$ and an $a \in L(S,x)$ which does belong to any E-component in $\mathbb{O}_S^{L(S,d)}$.

Fix such an x. In $L(S,x)$, let u be the set of all a which do not belong any E-component in $\mathbb{O}_S^{L(S,d)}$. So, $u \in \mathbb{O}_S^{L(S,x)}$.

As in Harrington's proof of the Silver dichotomy,

$$(u,u) \Vdash_{\mathbb{O}_S^{L(S,d)} \times \mathbb{O}_S^{L(S,d)}} \neg(\tau_L \, E \, \tau_R).$$

[The proof makes an appeal to ${}_2\mathbb{O}_S^{L(S,x)}$]

This gives an injection of $\mathbb{R}$ into $\mathbb{R}/E$.

This injection is obtained uniformly from S, x, and an enumeration of the dense sets of $\mathbb{O}_S^{L(S,x)}$ in $L(S,x)$.

Returning to the case $\{E_\alpha\}_{\alpha<\kappa}$, it suffices to find a single $x \in \mathbb{R}$ which works for all $\alpha < \kappa$. (For each $\alpha < \kappa$, almost all $x \in \mathcal{D}$ work).

- Pick a real $a \notin \mathrm{OD}_S$. For each $\alpha < \kappa$, a must fall under Case 2.
- There are only countably many distinct values for $[a]_{E_\alpha}$.

The last fact follows from: each $[a]_{E_\alpha}$ is a countable $\mathrm{OD}_{S,a}$ set and hence $[a]_{E_\alpha} \subseteq \mathrm{HOD}_{S,a}$. But $(2^\omega)^{\mathrm{HOD}_{S,a}}$ is countable, and a wellordered sequence of reals is countable.

For these countably many α giving distinct $[a]_{E_\alpha}$, intersect the corresponding cones.

Note that if $[a]_{E_\alpha} = [a]_{E_\beta}$, then x works for E_α iff x works for E_β.

This produces an $x \in \mathbb{R}$ which works (satisfies Case 2) for all $\alpha < \kappa$.

Upper Bound

Fact

Assume $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Let* $\kappa \in On$ *and* $\{E_\alpha\}_{\alpha<\kappa}$ *a sequence of smooth equivalence relations on* $\mathbb{R}$ *with all classes countable. Then* $\sqcup_{\alpha<\kappa}\mathbb{R}/E_\alpha$ *injects into* $\mathbb{R}\times\kappa$.

Sketch of proof:

Hjorth prooved an extension of the Harrington-Kechris-Louveau dichotomy:

Theorem (Hjorth)

Assume $\mathsf{ZF} + \mathsf{AD}^+$. *Let* E *be an equivalence relation on* $\mathbb{R}$. *Then either:*

1.) There is a wellordered separating family for E.

2.) There is a $\phi\colon \mathbb{R}\to\mathbb{R}$ *such that* $x E_0 y$ *iff* $\phi(x) E \phi(y)$.

Let (S, φ) be a code for E. There are again two cases.

Case 1. For all $x \in \mathbb{R}$ and all $a, b \in L(S, x)$ there is a $C \in \mathbb{O}_S^{L(S,x)}$ which is E-invariant with $a \in c$, $b \notin C$.

Using the wellfoundedness of $\prod_{\mathcal{D}} \omega_1 / \nu$ we get a separating family for E.

Case 2. There is an $x \in \mathbb{R}$ and $a, b \in L(S, x)$ with $\neg(aE\,b)$ such that there is no E-invariant $C \in \mathbb{O}_S^{L(S,x)}$ with $a \in C$, $b \notin C$.

In this case one produces an embedding of $\mathbb{R}/E_0$ into $\mathbb{R}/E$.

Let now $\{E_\alpha\}_{\alpha<\kappa}$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with countable classs.

- Let S be an ∞-Borel code for $\{E_\alpha\}_{\alpha<\kappa}$, which uniformly gives codes for the E_α.
- For each $\alpha < \kappa$, Case 2 cannot hold. The argument of Case 1 gives uniformly in α an injection $\Phi_\alpha \colon \mathbb{R}/E_\alpha \to \mathcal{P}(\delta)$ where $\delta = \prod_{\mathcal{D}} \omega_1/\nu$.
- We get an injection to $\kappa \times \mathbb{R}$ by using the following uniformization result.

Fact

Assume $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. *Let* $\delta \in On$, *X a set. Suppose* $R \subseteq \mathcal{P}(\delta) \times X$ *with all sections countable. Then R has a uniformization.*

Proof.

R is OD from a set S of ordinals. Each section R_T is thus OD from a set of ordinals. Then (case $X = \mathbb{R}$) each section has an $\mathrm{OD}_{S,T}$ ∞-Borel code. This shows $R_T \subseteq \mathrm{HOD}_{S,T}$.

□

Non-countable case

Now consider $X \subseteq \omega_1^{<\omega_1}$ and assume $\mathbb{R} \sqcup \omega_1$ injects into X.

Write $X = X_0 \sqcup \omega_1$, and let $X_0^\alpha = \{f \in X_0 \colon \sup(f) = \alpha\}$.

Let E_α be the equivalence relation on $\mathrm{WO}^\alpha = \{x \in (\mathrm{WO})^\omega \colon \sup |x(n)| = \alpha\}$ given by $x E_\alpha y$ iff $x, y \in \mathrm{WO}^\alpha$ and code the same set $A \in X_0^\alpha$ or $x, y \notin \mathrm{WO}^\alpha$ or don't code a set in X_0^α.

Each $\mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R}$ or is countable.

Let $A = \{\alpha \colon |\mathbb{R}/E_\alpha| = \omega\}$, $B = \omega_1 - A$.

For $\alpha \in A$, the $\mathbb{R}/E_\alpha$ are uniformly wellorderable using an ∞-Borel code for the sequence $\{E_\alpha\}_{\alpha<\omega_1}$.

If B is countable, then $X \approx \omega_1 \sqcup \mathbb{R}$, which is a union of smooth quotients.

If B is uncountable, this gives a bijection between X and $\sqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$.