On the preservation of very large cardinals under class forcing

Joan Bagaria

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Large cardinals are preserved by small forcing

Theorem (Levy-Solovay 1967)

All usual large cardinals are preserved by small (i.e., of size less than the cardinal) forcing notions. E.g., inaccessible, measurable, supercompact, etc.

Large cardinals are destroyed by big forcing

Any uncountable cardinal can be easily destroyed by some big forcing notion, e.g., by collapsing it.

And the inaccessibility of any given cardinal κ can be easily destroyed without collapsing any cardinals, e.g., by adding κ-many subsets of ω.

So, the general question is: What (big) forcing notions do preserve large cardinals?

For instance, does blowing up the power-set of κ preserve the large cardinal properties of κ?

Making a large cardinal indestructible

If the GCH holds below a measurable cardinal κ, then the standard forcing $\mathbb{P}$ that adds κ^{++}-many subsets of κ destroys the measurability of κ.

The forcing $\mathbb{P}$ is $< \kappa$-directed closed.

Richard Laver (1978): If κ is a supercompact cardinal, then there is a forcing notion (the *Laver preparation*) that preserves the supercompactness of κ and makes it indestructible under further $< \kappa$-directed closed forcing.

Usuba (2018) shows that if κ is a strongly compact cardinal, then there is a forcing extension in which the strong compactness of κ is indestructible under $\mathrm{Add}(\kappa, \delta)$, for any δ.

Preserving Σ_3-correct cardinals

If κ is supercompact, then $V_\kappa \preceq_{\Sigma_2} V$. Hence, after the Laver preparation forcing,

$$V[G]_\kappa \preceq_{\Sigma_2} V[G]$$

for every V-generic filter $G \subseteq \mathbb{P}$, whenever $\mathbb{P}$ is $< \kappa$-directed closed.

However, a similar Laver-indestructibility result for Σ_3-correct cardinals, and in particular for extendible cardinals, is not possible.

Definition

A cardinal κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, for some μ, such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Theorem (B-Hamkins-Tsaprounis-Usuba 2015)

Suppose that $V_\kappa \prec_{\Sigma_2} V_\lambda$ *and* $G \subseteq \mathbb{P}$ *is a* V*-generic filter for nontrivial strategically* $<\kappa$*-closed forcing* $\mathbb{P} \in V_\eta$*, where* $\eta \leqslant \lambda$ *. Then for every* $\theta \geqslant \eta$*,*

$$V_\kappa = V[G]_\kappa \not\prec_{\Sigma_3} V[G]_\theta.$$

In particular, every extendible cardinal κ is destroyed by any nontrivial strategically $<\kappa$-closed set forcing.

However, extendible cardinals, and even stronger large cardinal principles, implying Σ_n-correctness, $n \geqslant 3$, are preserved by suitable class-forcing iterations.

$C^{(n)}$-extendible cardinals

For each $n < \omega$, let $C^{(n)}$ be the Π_n-definable closed unbounded proper class of ordinals α that are Σ_n-correct, i.e., such that

$$V_\alpha \preceq_{\Sigma_n} V.$$

Definition

A cardinal κ is $C^{(n)}$-extendible (for $n \geqslant 1$) if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$.

A cardinal κ is extendible iff it is $C^{(1)}$-extendible.

$C^{(n)}$-extendible cardinals and Vopěnka's Principle

Recall that Vopěnka's Principle (VP) is the schema asserting that for every (definable) proper class of structures of the same type there exist distinct A and B in the class with an elementary embedding $j : A \to B$.

Theorem (B. 2012)

VP(Π_{n+1}), *namely* VP *restricted to classes of structures that are* Π_{n+1}*-definable, is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal. Hence* VP *is equivalent to the existence of a* $C^{(n)}$*-extendible cardinal for each* $n \geqslant 1$*.*

Brooke-Taylor (2011) shows that VP is indestructible under ORD-length iterations with Easton support of increasingly directed-closed forcing notions (*without the need of any preparatory forcing!*).

Preserving $C^{(n)}$-extendible cardinals under class forcing

Question

What ORD*-length forcing iterations preserve extendible and $C^{(n)}$-extendible cardinals?*

The problem is how to lift (a proper class of) elementary embeddings of the form

$$j : V_\lambda \to V_\mu$$

witnessing the $C^{(n)}$-extendibility of $crit(j)$, to

$$j : V_\lambda[G_\lambda] \to V_\mu[G_\mu]$$

where G is $\mathbb{P}$-generic over V.

The following is a joint work with **Alejandro Poveda**

Magidor's characterization of supercompact cardinals

Theorem (Magidor 1971)

For a cardinal δ, the following statements are equivalent:

1. δ *is a supercompact cardinal.*
2. *For every* $\lambda > \delta$ *there exist ordinals* $\bar{\delta} < \bar{\lambda} < \delta$ *and an elementary embedding* $j : V_{\bar{\lambda}} \longrightarrow V_{\lambda}$ *such that:*
    - $\mathrm{cp}(j) = \bar{\delta}$ *and* $j(\bar{\delta}) = \delta$.

Magidor's characterization of supercompact cardinals

Theorem (Magidor 1971)

For a cardinal δ, the following statements are equivalent:

1. δ *is a supercompact cardinal.*
2. *For every $\lambda > \delta$ in $C^{(1)}$ and for every $a \in V_\lambda$, there exist ordinals $\bar{\delta} < \bar{\lambda} < \delta$ and there exist some $\bar{a} \in V_{\bar{\lambda}}$ and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:*
   - $cp(j) = \bar{\delta}$ *and* $j(\bar{\delta}) = \delta$.
   - $j(\bar{a}) = a$.
   - $\bar{\lambda} \in C^{(1)}$.

Σ_n-supercompact cardinals

Definition

If $\lambda > \delta$ is in $C^{(n)}$, then we say that δ is λ-Σ_n-supercompact if for every $a \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $cp(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda} \in C^{(n)}$.

We say that δ is Σ_n−supercompact if it is λ-Σ_n-supercompact for every $\lambda > \delta$ in $C^{(n)}$.

Theorem (Poveda 2018, Boney 2018)

A cardinal δ is Σ_{n+1}-supercompact if and only if it is $C^{(n)}$-extendible.

In particular, a cardinal is extendible if and only if it is Σ_2-supercompact.

Thus, δ is $C^{(n)}$-extendible if and only if for a proper class of λ in $C^{(n+1)}$, for every $\alpha < \lambda$ there exist $\bar{\delta}, \bar{\alpha} < \bar{\lambda}$ and an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_{\lambda}$ such that:

- $cp(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{\alpha}) = \alpha$.
- $\bar{\lambda} \in C^{(n+1)}$.

Lifting λ-Σ_n-supercompact embeddings

We make use of this characterization of $C^{(n)}$-extendibility to show that many ORD-length forcing iterations $\mathbb{P}$ preserve $C^{(n)}$-extendible cardinals.

For this, one lifts ground model embeddings $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ witnessing the λ-Σ_{n+1}-supercompactness of δ to embeddings $j : V[G]_{\bar{\lambda}} \longrightarrow V[G]_\lambda$ verifying in $V[G]$ the same property.

Key point: The cardinals λ for which this will be possible need to be sufficiently correct.

$\mathbb{P}$-reflecting cardinals

Let $\mathbb{P}$ be an ORD-length iteration.

Let us call a cardinal λ is $\mathbb{P}$-reflecting if $\mathbb{P}$ forces that $V[\dot{G}]_\lambda \subseteq V_\lambda[\dot{G}_\lambda]$.
(Hence, if G is $\mathbb{P}$-generic over V, then $V[G]_\lambda = V_\lambda[G_\lambda]$.)

A second reflection property of λ that will be required in our arguments is that

$$\langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$$

for some big-enough k.

Let $C^{(k)}_{\mathbb{P}}$ be the closed and unbounded class of such cardinals λ.

In the case $\mathbb{P}$ is Γ_m-definable[1] for some $m \geqslant 1$, where Γ is either Σ or Π, then

Proposition

The class $C^{(k)}_{\mathbb{P}}$ *is*

$$\begin{array}{ll} \Pi_{m+1}\text{-definable}, & \text{if } k = 1 \text{ and } \mathbb{P} \text{ is } \Gamma_m\text{-definable}. \\ \Pi_{m+k-1}\text{-definable}, & \text{if } k \geqslant 2 \text{ and } \mathbb{P} \text{ is } \Gamma_m\text{-definable}. \end{array}$$

[1] When we say that a forcing notion $\mathbb{P}$ is Γ_m-definable, we mean that the ordering relation $\leqslant_{\mathbb{P}}$ is Γ_m-definable, hence the set of conditions is also Γ_m-definable.

A key lemma

The following is a key lemma:

Lemma

Suppose $\mathbb{P}$ *is a definable iteration. If* κ *is a* $\mathbb{P}$*-reflecting cardinal in* $C^{(k)}_{\mathbb{P}}$*, then* $\mathbb{P}$ *forces* $V[\dot{G}]_\kappa \prec_{\Sigma_k} V[\dot{G}]$.

Thus, we give such cardinals a name:

Definition

A cardinal κ is $\mathbb{P}$-Σ_k-reflecting if it is $\mathbb{P}$-reflecting and belongs to $C^{(k)}_{\mathbb{P}}$.

$\mathbb{P}$-Σ_n-supercompactness

Definition

If $\mathbb{P}$ is a definable iteration, then we say that a cardinal δ is $\mathbb{P}$-Σ_n-supercompact if there exists a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals, and for every such cardinal $\lambda > \delta$ and every $a \in V_\lambda$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and $\bar{a} \in V_{\bar{\lambda}}$, and there exists an elementary embedding $j : V_{\bar{\lambda}} \longrightarrow V_\lambda$ such that:

- $cp(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.
- $j(\bar{a}) = a$.
- $\bar{\lambda}$ is $\mathbb{P}$-Σ_n-reflecting.

Proposition

If $\mathbb{P}$ is a Γ_m-definable iteration, some $m \geqslant 1$, then assuming there is a proper class of $\mathbb{P}$-Σ_{n+1}-reflecting cardinals,

1. *Every $\mathbb{P}$-Σ_{n+1}-supercompact cardinal is $C^{(n)}$-extendible.*
2. *Every $C^{(n)}$-extendible cardinal is $\mathbb{P}$-Σ_{n+1}-supercompact, in the case $m = 1$.*
3. *Every $C^{(m+n-1)}$-extendible cardinal is $\mathbb{P}$-Σ_{n+1}-supercompact, in the case $m \geqslant 2$.*

In particular, if $\mathbb{P}$ is a Γ_1-definable iteration and there exists a proper class of $\mathbb{P}$-Σ_n-reflecting cardinals, a cardinal is $C^{(n)}$-extendible if and only if it is $\mathbb{P}$-Σ_{n+1}-supercompact.

Suitable iterations

Definition

A forcing iteration $\mathbb{P}$ is suitable if it is the direct limit of an Easton support iteration[2] $\langle \mathbb{P}_\lambda; \dot{\mathbb{Q}}_\lambda : \lambda < \mathrm{ORD} \rangle$ such that for each λ,

1. If λ is an inaccessible cardinal, then $\mathbb{P}_\lambda \subseteq V_\lambda$.
2. There is some $\theta > \lambda$ such that

$$\Vdash_{\mathbb{P}_\nu} \text{“}\dot{\mathbb{Q}}_\nu \text{ is } \lambda\text{-directed closed”}$$

   for all $\nu \geqslant \theta$.

Recall that a partial ordering $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ and q are compatible.

[2] Recall that an Easton support iteration is a forcing iteration where direct limits are taken at inaccessible stages and inverse limits elsewhere.

Main preservation theorem

Theorem

Suppose $m, n \geqslant 1$ *and* $m \leqslant n+1$. *Suppose* $\mathbb{P}$ *is a weakly homogeneous* Γ_m*-definable suitable iteration and there exists a proper class of* $\mathbb{P}$*-*Σ_{n+1}*-reflecting cardinals. If* δ *is a* $\mathbb{P}$*-*Σ_{n+1}*-supercompact cardinal, then*

$$\Vdash_{\mathbb{P}} \text{“}\, \delta \textit{ is } C^{(n)}\textit{-extendible”}.$$

Corollary

Suppose $n \geqslant 1$, $\mathbb{P}$ *is a weakly homogeneous* Γ_1*-definable suitable iteration,* δ *is a* $C^{(n)}$*-extendible cardinal, and there is a proper class of* $\mathbb{P}$-Σ_{n+1}*-reflecting cardinals. Then*

$$\Vdash_{\mathbb{P}} \text{"}\delta \textit{ is } C^{(n)}\textit{-extendible"}.$$

Proposition

If ORD *is* Π_{m+n}*-Mahlo (i.e., every* Π_{m+n}*-definable club proper class of ordinals contains an inaccessible cardinal), in case* $\Gamma = \Sigma$ *or* $n > 1$*, or is* Π_{m+2}*-Mahlo in case* $\Gamma = \Pi$ *and* $n = 1$*, then the class of* $\mathbb{P}$-Σ_{n+1}*-reflecting cardinals is proper.*

Corollary

Suppose that $1 \leqslant m, n$ *with* $m \leqslant n+1$, $\mathbb{P}$ *is a weakly homogeneous* Γ_m*-definable suitable iteration, and* δ *is a* $\mathbb{P}$-Σ_{n+1}*-supercompact cardinal. If* ORD *is* Π_{m+n}*-Mahlo (case* $\Gamma = \Sigma$ *or* $n > 1$*), or* Π_{m+2}*-Mahlo (case* $\Gamma = \Pi$ *and* $n = 1$*), then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \textit{ is } C^{(n)}\textit{-extendible”}.$$

Somme applications

Preserving VP level-by-level

Theorem (Brooke-Taylor 2011)

Let $\mathbb{P}$ be a definable suitable iteration. If VP *holds in* V, *then* VP *holds in* $V^{\mathbb{P}}$.

Theorem

Let $n, m \geqslant 1$ *be such that* $m \leqslant n + 1$, *and let* $\mathbb{P}$ *be a weakly thomogeneous* Γ_m*-definable suitable iteration. Then,*

1. *If* $\Gamma = \Sigma$ *or* $n > 1$, *and* $\text{VP}(\Pi_{m+n})$ *holds, then* $\text{VP}(\Pi_{n+1})$ *holds in* $V^{\mathbb{P}}$.
2. *If* $\Gamma = \Pi$ *and* $n = 1$, $\text{VP}(\Pi_{m+1})$ *holds, and ORD is* Π_{m+2}*-Mahlo, then* $\text{VP}(\Pi_2)$ *holds in* $V^{\mathbb{P}}$.

$C^{(n)}$-extendible cardinals and the GCH

Let $\mathbb{P} = \langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathrm{ORD} \rangle$ be the standard Jensen's proper class iteration for forcing the global GCH. Namely, the direct limit of the iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is an uncountable cardinal", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha^+, 1)$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial", otherwise.

$\mathbb{P}$ is weakly homogeneous, suitable, and Π_1-definable.

Theorem (Tsaprounis 2013)

Forcing with $\mathbb{P}$ preserves $C^{(n)}$-extendible cardinals.

Changing the power-set function on regular cardinals

Recall: A class function E from the class REG of infinite regular cardinals to the class of cardinals is an Easton function if:

1. $\mathrm{cf}(E(\kappa)) > \kappa$, for all $\kappa \in \mathsf{REG}$
2. If $\kappa \leqslant \lambda$, then $F(\kappa) \leqslant F(\lambda)$

Let $\mathbb{P}_E = \lim\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha :\, \alpha \in \mathsf{ORD}\rangle$ be the forcing iteration with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is a regular cardinal", then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha, E(\alpha))$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

If the GCH holds in the ground model, then $\mathbb{P}_E$ preserves all cardinals and cofinalities and forces that $2^\kappa = E(\kappa)$ for every regular cardinal κ.

$\mathbb{P}_E$ is suitable and weakly homogeneous.

Changing the power-set function on regular cardinals

Theorem

If E is a Δ_2-definable Easton function, then $\mathbb{P}_E$ preserves $C^{(n)}$-extendible cardinals, all $n \geqslant 1$. More generally, if E is a Π_m-definable Easton function ($m > 1$) and λ is $C^{(m+n-1)}$-extendible, then $\mathbb{P}_E$ forces that λ is $C^{(n)}$-extendible, all $n \geqslant 1$ such that $m \leqslant n + 1$.

The theorem is sharp: If κ is the least $C^{(n)}$-extendible cardinal, then the Easton function E that sends $\aleph_0$ to κ and every uncountable regular cardinal λ to $\max\{\lambda^+, \kappa\}$ is Π_{n+2}-definable and destroys κ being inaccessible. In the case $n = 1$ this gives in fact an example of a Π_2-definable Easton function E such that $\mathbb{P}_E$ destroys an extendible cardinal.

Corollary

For every definable Easton function E the class forcing $\mathbb{P}_E$ preserves VP.

Corollary

If VP holds in V, then in some class forcing extension of V that preserves VP, for every definable Easton function E there is a further class forcing extension that preserves VP and where $2^\kappa = E(\kappa)$ for every infinite regular cardinal κ.

Forcing V "far" from HOD

Let $\mathbb{C} = \langle \mathbb{P}_\alpha ; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathrm{ORD} \rangle$ be the Easton support iteration where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Coll}}(\alpha, \alpha^+)$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

Theorem

Forcing with $\mathbb{C}$ preserves $C^{(n)}$-extendible cardinals (hence also VP) and forces $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$, for every regular cardinal λ.

Note: Forcing $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$, for some singular cardinal λ, while preserving some extendible cardinal smaller than λ would refute Woodin's HOD Conjecture.

Forcing further disagreement between V and HOD

Let K be a function on the class of infinite cardinals such that $K(\lambda) > \lambda$, for every λ, and K is increasingly monotone. Let $\mathbb{P}_K$ be the direct limit of an iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha : \alpha \in \mathsf{ORD} \rangle$ with Easton support where $\mathbb{P}_0$ is the trivial forcing and for each ordinal α, if $\Vdash_{\mathbb{P}_\alpha}$ "α is regular" then $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Coll}}(\alpha, K(\alpha))$", and $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is trivial" otherwise.

$\mathbb{P}_K$ preserves all inaccessible cardinals that are closed under K. Moreover, for each α such that $\Vdash_{\mathbb{P}_\alpha}$ "α is regular", the remaining part of the iteration after stage α is α-closed, hence it preserves α. Also, if K is Π_m-definable ($m \geqslant 1$), then $\mathbb{P}_K$ is also Π_m-definable.

Theorem

If K is Δ_2-definable, then $\mathbb{P}_K$ preserves $C^{(n)}$-extendible cardinals, all $n \geqslant 1$. More generally, if K is Π_m-definable ($m > 1$) and λ is $C^{(m+n-1)}$-extendible, then $\mathbb{P}_K$ forces that λ is $C^{(n)}$-extendible, all $n \geqslant 1$ such that $m \leqslant n + 1$. Moreover, $\mathbb{P}_K$ forces

$$(\lambda^+)^{\mathsf{HOD}} \leqslant K(\lambda) < \lambda^+$$

for all infinite regular cardinals λ.

The function K may be taken so that $\mathbb{P}_K$ destroys many singular cardinals in HOD while preserving extendible cardinals.
For example, let K be such that $K(\lambda)$ is the least singular cardinal in HOD greater than λ, i.e., $K(\lambda) = (\lambda^{+\omega})^{\mathsf{HOD}}$. Then, K is Δ_2-definable, and we have the following.

Corollary

$\mathbb{P}_K$ *preserves extendible cardinals and forces*

$$(\lambda^{+\omega})^{\mathsf{HOD}} < \lambda^+$$

for every regular cardinal λ.

Strong diamonds

A sequence $\langle \mathcal{A}_\alpha : \alpha \in \kappa^+ \rangle$ is a $\diamondsuit^+_{\kappa^+}$ sequence if $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^{\leqslant\kappa}$ and for every $A \subseteq \kappa^+$ there is a club $C \subseteq \kappa^+$ such that if $\alpha \in \mathrm{Lim}(C)$, then $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$.

Cummings-Foreman-Magidor (2001) show that assuming $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$, there is a κ^+-closed and κ^{++}-cc forcing notion $\mathbb{D}_\kappa$ that forces the existence of a $\diamondsuit^+_{\kappa^+}$ sequence. The forcing is Π_1-definable.

Let $\mathbb{D}$ be the standard Easton support iteration of the forcings $\mathbb{D}_\kappa$, any cardinal κ. $\mathbb{D}$ is a weakly homogeneous, suitable, and Π_1-definable iteration.

Theorem

If the GCH holds, then forcing with $\mathbb{D}$ preserves $C^{(n)}$-extendible cardinals and forces $\diamondsuit^+_{\kappa^+}$ for every cardinal κ. Hence if VP holds, then one can force $\diamondsuit^+_{\kappa^+}$ for every cardinal κ, while preserving VP.

Weak squares

Definition

Let $\mu \leqslant \kappa$ be infinite cardinals. A $\square_{\kappa,\mu}$-sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \mathrm{Lim} \cap \kappa^+ \rangle$ such that for all $\alpha \in \mathrm{Lim} \cap \kappa$:

(a) $\mathcal{C}_\alpha$ is a non-empty family of club subsets of α with $|\mathcal{C}_\alpha| \leqslant \mu$.

(b) If $\mathrm{cof}(\alpha) < \kappa$, then every element of $\mathcal{C}_\alpha$ has order-type $< \kappa$.

(c) If $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{Lim}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.

We say that $\square_{\kappa,\mu}$ holds if there exists a $\square_{\kappa,\mu}$-sequence.

Cummings-Foreman-Magidor (2001) show that if κ is supercompact and $\kappa \leqslant \mathrm{cof}(\lambda) < \lambda$, then one can force $\square_{\lambda,\mathrm{cof}(\lambda)}$ while preserving the supercompactness of κ.

Weak squares

Theorem

There is a class forcing iteration that preserves $C^{(n)}$-extendible cardinals, all $n < \omega$, and forces that for every uncountable cardinal λ, if $K(\lambda)$ is the first singular cardinal of cofinality λ^+, then $\square_{K(\lambda),\lambda^+}$ holds.

Corollary

If VP holds, then there is a class forcing iteration that preserves VP and forces $\square_{\lambda,\mathrm{cof}(\lambda)}$, for a proper class of singular cardinals λ.