RELATIVIZING TO AN ENUMERATION ORACLE

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In the "*Celestial Emporium of Benevolent Knowledge*... it is written that animals are divided into

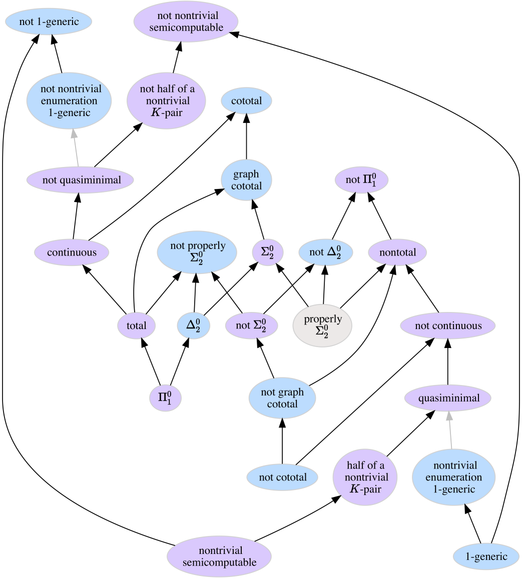
(a) those that belong to the Emperor,
(b) embalmed ones,
(c) those that are trained,
(d) suckling pigs,
(e) mermaids,
(f) fabulous ones,
(g) stray dogs,
(h) those that are included in this classification,
(i) those that tremble as if they were mad,
(j) innumerable ones,
(k) those drawn with a very fine camel's hair brush,
(l) others,
(m) those that have just broken a flower vase,
(n) those that resemble flies from a distance."

Jorge Luis Borges, "The Analytical Language of John Wilkins"

Towards a taxonomy of the enumeration degrees

I am interested in identifying and understanding natural subclasses of the enumeration degrees.

We know some:



Towards a taxonomy of the enumeration degrees

How might we identify interesting subclasses?

(1) Consider the degrees containing some specific kind of set, e.g., Σ^0_2 degrees, 1-generic degrees, or degrees containing sets of the form $A \oplus \overline{A}$ (the *total* degrees).

(2) Consider classes of degrees with natural (first order) definitions in the partial order.

(3) Consider the degrees of points in (sufficiently effective) nice topological spaces, e.g., the *continuous degrees*, the degrees of points in $\mathcal{C}[0,1]$. (See Kihara, Ng, and Pauly for more examples.)

(4) Consider classes of degrees that "behave similarly" to the Turing degrees.

Note. These approaches are not mutually exclusive (some classes have characterizations of all four types!), and are unlikely to be exhaustive.

Which e-degrees behave like Turing degrees?

For this talk, we are interested in the (4)th approach.

Relativizing to an enumeration oracle

- Many important notions in computability theory are expressed in terms of c.e. sets.
- Of course, when we relativize these notions to a Turing oracle X, we use X-c.e. sets.
- Relativizing to an enumeration oracle A is straightforward: simply replace "X-c.e." with "c.e. in every enumeration of A" (i.e., $\leqslant_e A$).

Sometimes, results in the Turing degrees relativize to enumeration oracles. Frequently, they do not, *but that's okay!*

Goal. Define new (or characterize known) classes of enumeration degrees as those relative to which a given theorem remains true.

Preliminaries: the enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B.

Definition. $A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e\rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

Preliminaries: the total enumeration degrees

Proposition. $A \leqslant_T B$ iff $A \oplus \overline{A}$ is B-c.e. iff $A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound (and even the jump).

Definition. The *total degrees* are the image of the Turing degrees under this embedding (i.e., they are the enumeration degrees that contain a set of the form $A \oplus \overline{A}$).

It is easy to see that there are nontotal enumeration degrees. In fact, a sufficiently generic $A \subseteq \omega$ has nontotal degree.

Case Study: PA relative to $\langle A \rangle$[1]

[1] **Notation.** We use $\langle A \rangle$ to indicate that we are viewing $A \subseteq \omega$ as an enumeration oracle.

Relativizing the PA degrees

Recall. Let $A \subseteq \omega$. We call $U \subseteq 2^\omega$ a *$\Sigma^0_1[A]$ class* if there is an A-c.e. set of strings W, such that $U = [W]$,

$$[W] = \{X \in 2^\omega : (\exists \sigma \in W)\ X \succeq \sigma\}.$$

Let's translate this to the enumeration oracle case:

Definition. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a *$\Sigma^0_1\langle A\rangle$ class* if there is a set of strings $W \leqslant_e A$, such that $U = [W]$.

A *$\Pi^0_1\langle A\rangle$ class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

Note that a $\Pi^0_1\langle A \oplus \overline{A}\rangle$ class is just a $\Pi^0_1[A]$ class in the usual sense.

Definition. $\langle B\rangle$ is *PA relative to $\langle A\rangle$* if in every nonempty $\Pi^0_1\langle A\rangle$ class P there is an $X \in P$ such that $X \oplus \overline{X} \leqslant_e B$.

We treat members of P as total objects, but it doesn't matter!

The PA degrees relative to $\langle B \rangle$

Do the PA degrees relative to $\langle A \rangle$ behave like they do in the Turing degrees? More specifically:

(a) Can $\langle A \rangle$ be PA relative to $\langle A \rangle$? **Yes!**

(b) Does $\langle B \rangle$ being PA relative to $\langle A \rangle$ imply that $B \geqslant_e A$? **No!**

(c) Must there be a nonempty $\Pi^0_1\langle A \rangle$ class P such that every $B \in P$ is PA relative to $\langle A \rangle$, i.e., a "universal" $\Pi^0_1\langle A \rangle$ class? **No!**

So "PA relative to $\langle A \rangle$" can behave very differently from our Turing-centric expectations.

But this is good for our purposes: we can ask for which class of enumerations oracles do each of these properties hold.

$\langle$self$\rangle$-PA degrees

(a) Can $\langle A \rangle$ be PA relative to $\langle A \rangle$? **Yes!**

In other words, the fact that there is an infinite Π^0_1 (even computable) tree $T \subseteq 2^{<\omega}$ with no computable paths (Kleene 1952) *does not relativize* to an arbitrary enumeration degree.

Definition (M., Soskova)

We say that A is *$\langle$self$\rangle$-PA* if $\langle A \rangle$ is PA relative to $\langle A \rangle$.

$\langle\text{self}\rangle$-PA degrees exist

Proposition

There is a $\langle\text{self}\rangle$-PA set A.

Proof.

Fix a sequence of sets $\langle X_e \rangle_{e\in\omega}$ such that X_{e+1} is PA relative to X_e. (Note that $X_e \equiv_T \bigoplus_{i\leqslant e} X_i$.) We build A in stages, where A_s will be determined on finitely many columns plus finitely much more.

At stage $s = 2e$, we copy $X_e \oplus \overline{X_e}$ to (the as of yet undetermined part of) the eth column of A_s to get A_{s+1}.

At stage $s = 2e + 1$, we try to extend A_s by finitely much to force the eth $\Pi^0_1\langle A\rangle$ class, $P_e\langle A\rangle$, to be empty. If we can, *great!*

If we can't, let Z be the set where all undetermined positions of A_s are replaced by 1s. Then $P\langle Z\rangle \subseteq P\langle A\rangle$ is a nonempty $\Pi^0_1[X_e]$ class. Therefore, it has an element below $X_{e+1} \oplus \overline{X_{e+1}}$, which will be coded into A (at stage $s + 1$). $\square$

Where are the $\langle\text{self}\rangle$-PA degrees?

With a little more care, we can show:

Proposition

1. If $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a countable *Scott ideal* (i.e., for all $X \in \mathcal{I}$ there is a $Y \in \mathcal{I}$ that is PA relative to X), then there is a $\langle\text{self}\rangle$-PA set A such that $X \oplus \overline{X} \leqslant_e A$ if and only if $X \in \mathcal{I}$.
2. X is PA relative to Y if and only if there is a $\langle\text{self}\rangle$-PA set A such that $Y \oplus \overline{Y} \leqslant_e A \leqslant X \oplus \overline{X}$.

Corollary. There is a Δ^0_2 $\langle\text{self}\rangle$-PA degree.

Proposition. If A is $\langle\text{self}\rangle$-PA, then $\mathcal{I} = \{X : X \oplus \overline{X} \leqslant_e A\}$ is a countable Scott ideal.

Proof. Fix $X \in \mathcal{I}$ and let P be a nonempty $\Pi^0_1[X]$ class containing only elements PA above X. Since P is also a $\Pi^0_1\langle A\rangle$ class, there is a $Y \in P$ such that $Y \oplus \overline{Y} \leqslant_e A$. So $Y \in \mathcal{I}$ and Y is PA relative to X. $\square$

Continuous degrees

(b) Does $\langle B \rangle$ being PA relative to $\langle A \rangle$ imply that $B \geqslant_e A$? **No!**

The sets for which this is *true* were called *PA bounded* in an upcoming paper of Ganchev, Kalimullin, M., Soskova. They turned out to be a familiar class:

Theorem (Franklin, Lempp, M., Schweber, Soskova)

As set $A \subseteq \omega$ has continuous degree if and only if $\langle B \rangle$ being PA relative to $\langle A \rangle$ implies that $B \geqslant_e A$.

Continuous degrees

Basic Facts (M. 2004).

- The *continuous degrees* are the degrees of points in computable metric spaces.
- They properly extend the Turing degrees (which are the degrees of points in 2^ω, ω^ω, and $\mathbb{R}$), and properly embed into the enumeration degrees in a natural way.
- Every continuous degree contains a point from $[0,1]^\omega$ (the Hilbert cube) and a point from $\mathcal{C}[0,1]$ (hence the name).

Instead of delving into the necessary computable analysis to define the continuous degrees properly, let's skip to some nice characterizations inside the enumeration degrees.

Definition (Andrews, Igusa, M., Soskova)

$A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ enumerates A. If there is a c.e. operator W such that $A = W^X$ for every $X \in P$, then A is *uniformly codable*.

Codability

Definition. $A \subseteq \omega$ is *codable* if there is a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $X \in P$ enumerates A.

Theorem (Andrews, Igusa, M., Soskova)

The following are equivalent for $A \subseteq \omega$:

- A has continuous enumeration degree,
- A is (uniformly) codable,
- A has *almost total* enumeration degree: whenever $\mathbf{b} \nleqslant \deg_e(A)$ is total, $\deg_e(A) \vee \mathbf{b}$ is also total.

Clearly, if A is codable, then whenever $\langle B\rangle$ is PA relative to $\langle A\rangle$, we have $B \geqslant_e A$. We have already admitted to the converse:

Theorem (Franklin, Lempp, M., Schweber, Soskova)

As set $A \subseteq \omega$ has continuous degree if and only if $\langle B\rangle$ being PA relative to $\langle A\rangle$ implies that $B \geqslant_e A$.

Aside: the nontotal continuous degrees

As we've said, the continuous degrees *properly* extend the Turing degrees. This is an essentially topological fact:

Remark. The fact that the Hilbert cube is not a countable union of subspaces of Cantor space is easily proved from the fact that there are nontotal continuous degrees in every cone, and visa versa (Kihara & Pauly and Mathieu Hoyrup).

There is also a nice connection to Scott ideals and PA degrees:

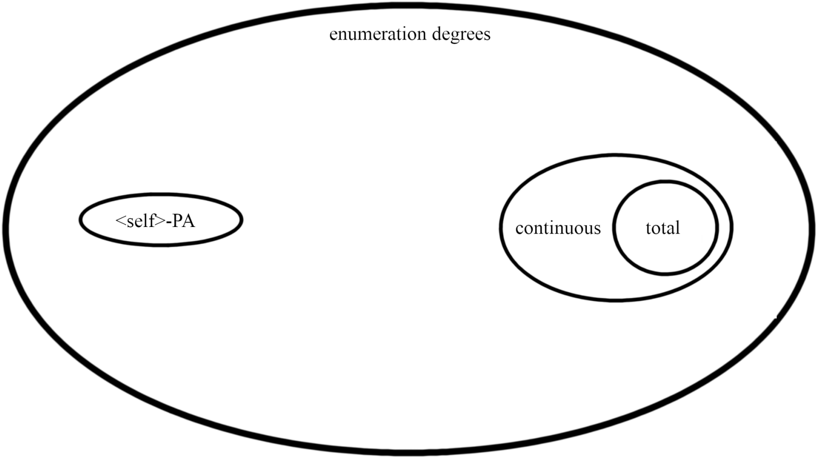
Theorem (M. 2004)

1. $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a countable *Scott ideal* if and only if there is a set A of nontotal continuous degree such that $\mathcal{I} = \{X : X \oplus \overline{X} \leqslant_e A\}$.
2. X is PA relative to Y if and only if there is a set A of nontotal continuous degree such that $Y \oplus \overline{Y} \leqslant_e A \leqslant X \oplus \overline{X}$.

Does this mean that the nontotal continuous degrees are related to the $\langle$self$\rangle$-PA degrees? *No!*

How does everything relate so far?

We will see that the ⟨self⟩-PA degrees are disjoint from the continuous degrees. So we have something like:



Universal $\Pi^0_1\langle A\rangle$ classes

(c) Must there be a nonempty $\Pi^0_1\langle A\rangle$ class P such that every $B \in P$ is PA relative to $\langle A\rangle$, i.e., a "universal" $\Pi^0_1\langle A\rangle$ class? **No!**

Question

For which enumeration oracles A is there a *universal* $\Pi^0_1\langle A\rangle$ *class*?

We have several partial results.

Universal $\Pi^0_1\langle A\rangle$ classes

Proposition (Ganchev, Kalimullin, M., Soskova)

If $A \subseteq \omega$ has continuous degree, then there is a universal $\Pi^0_1\langle A\rangle$ class.

Proof. Assume that the codability of A is witnessed by the nonempty $\Pi^0_1\langle A\rangle$ class P. Consider the nonempty $\Pi^0_1\langle A\rangle$ class

$$Q = \{X \oplus Y : X \in P \text{ and } Y \text{ is } \mathrm{DNC}_2 \text{ relative to } X\}.$$

Note that $B \in Q$ implies that B is PA relative to some $X \in P$. But $A \leqslant_e X \oplus \overline{X}$, so B is PA relative to $\langle A\rangle$. $\square$

Proposition (Ganchev, Kalimullin, M., Soskova)

If $A \subseteq \omega$ is $\langle$self$\rangle$-PA, then there is *no* universal $\Pi^0_1\langle A\rangle$ class.

Corollary (M., Soskova)

The $\langle$self$\rangle$-PA degrees are disjoint from the continuous degrees.

Universal $\Pi^0_1\langle A\rangle$ classes…or the lack thereof

Proposition (Ganchev, Kalimullin, M., Soskova)

If $A \subseteq \omega$ is $\langle\text{self}\rangle$-PA, then there is *no* universal $\Pi^0_1\langle A\rangle$ class.

Proof. If there were a universal $\Pi^0_1\langle A\rangle$ class P, then there would be an $X \in P$ such that $X \oplus \overline{X} \leqslant_e A$ and X is PA relative to $\langle A\rangle$. In that case, X would be PA relative to every Y such that $Y \oplus \overline{Y} \leqslant_e A$. In particular, X would be PA relative to X, which is impossible. $\square$

Actually, these are the only enumeration degrees we know of without a universal class.

Open Question

If $A \subseteq \omega$ is not $\langle\text{self}\rangle$-PA, must it have a universal class?

We suspect that the answer is *no!*

Low for PA

We (Franklin, Lempp, M., Schweber, Soskova) do have another source of enumeration degrees with universal classes.

Definition. An enumeration oracle $\langle A \rangle$ is *low for PA* if every PA degree is PA relative to $\langle A \rangle$. I.e., whenever X has PA degree and P is a nonempty $\Pi^0_1\langle A \rangle$ class, there is a $Y \in P$ such that $X \geqslant_T Y$.

Proposition. Assume that $\langle A \rangle$ is low for PA.

(i) A is c.e. or has has quasiminimal enumeraion degree.

(ii) DNC_2 is a universal $\Pi^0_1\langle A \rangle$ class.

Proof. (ii) is obvious. To see (i), assume that $A \geqslant_e Z \oplus \overline{Z}$, where Z is not computable. Then $\{Z\}$ is a $\Pi^0_1\langle A \rangle$ class. Take any PA degree X that does not compute Z. Then X does not compute any member of $\{Z\}$, so $\langle A \rangle$ is not low for PA. $\square$

Theorem. $\langle A \rangle$ is low for PA if and only if whenever P is a nonempty $\Pi^0_1\langle A \rangle$ class, there is a nonempty Π^0_1 subclass $Q \subseteq P$.

Low for PA

Proposition. If A is 1-generic, then $\langle A\rangle$ is low for PA.

Proof. Let $P\langle A\rangle$ be a nonempty $\Pi^0_1\langle A\rangle$ class. We claim that there is a prefix $\sigma \prec A$ such that $P\langle \sigma 1^\omega\rangle$ is nonempty. If so, then $Q = P\langle \sigma 1^\omega\rangle$ is a nonempty Π^0_1 class and $Q \subseteq P\langle A\rangle$.

So assume that A has no such prefix. Consider the c.e. set of strings $W = \{\tau \in 2^{<\omega} : P\langle \tau\rangle = \varnothing\}$. Because $P\langle A\rangle$ is nonempty, there is no prefix of A in W. However, by assumption, every $\sigma \prec A$ can be extended to a string $\sigma 1^m \in W$. So W is dense along A but contains no prefix of A, contradicting 1-genericity of A. □

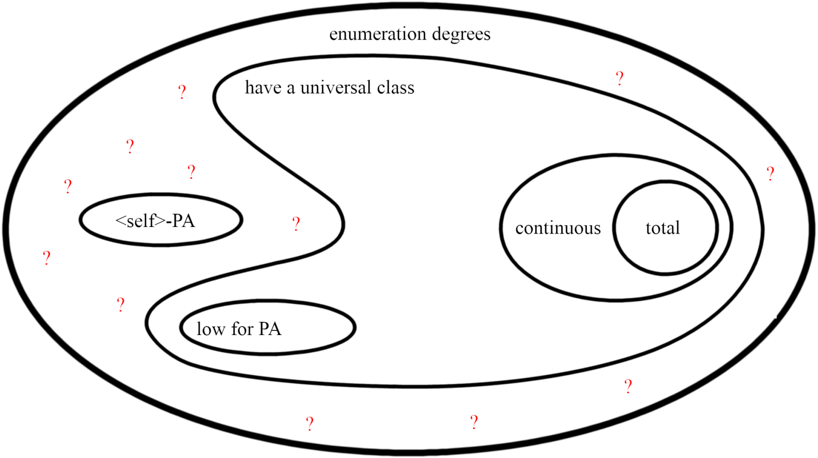
Not every quasiminimal degree is low for PA. Lagemann showed that sufficiently random enumeration oracles are quasiminimal. However:

Proposition. If A is Martin-Löf random, then $\langle A\rangle$ is *not* low for PA.

Open Question. If A is sufficiently random, is there necessarily a universal $\Pi^0_1\langle A\rangle$ class?

How do things relate now?

No quasiminimal degree is continuous, so our picture looks something like this. But as mentioned above, the indicated region may be empty.



Case Study: Relativizing the Jump

What is the enumeration jump?

In the Turing degrees, the jump X' can be viewed as the uniform join of all X-c.e. sets. Let's relativize this to an enumeration oracle.

Definition

Let $K_A = \bigoplus_{e\in\omega} \Gamma_e\langle A\rangle$, where Γ_e is the eth enumeration operator.

In other words, K_A is the uniform join of the sets that are $\leqslant_e A$. Is this a good analogue of the jump? **No!**

$$\text{It is easy to see that } A \equiv_e K_A.$$

What went wrong? The proof that $X' \nleqslant_T X$ uses that the *complement* $\overline{X'}$ is not X-c.e., so maybe the better analogue is:

Definition

The *skip* of $A \subseteq \omega$ is the set $A^\Diamond = \overline{K_A}$.

Now we have $A^\Diamond \nleqslant_e A$, for all $A \subseteq \omega$.

What is the enumeration jump?

The skip agrees with the Turing jump on the total degrees. Moreover, it has some of the properties that we expect from the jump. For example (Andrews, Ganchev, Kuyper, Lempp, M., A. Soskova, and M. Soskova 2019):

- $A \leqslant_e B$ if and only if $A^\Diamond \leqslant_1 B^\Diamond$.
- Degree Invariance: If $A \equiv_e B$, then $A^\Diamond \equiv_e B^\Diamond$.
- Skip Inversion: For every $S \geqslant_e \varnothing^\Diamond$, there is a set A such that $A^\Diamond \equiv_e S$.

However, the skip is not the accepted enumeration jump:

Definition (Cooper 1984)

The *enumeration jump* of $A \subseteq \omega$ is the set $J_e(A) = K_A \oplus \overline{K_A}$.

Cooper intended to use the skip as the enumeration jump until his student McEvoy noted that it isn't always the case that $A^\Diamond \geqslant_e A$.

The cotatal degrees

In fact, it follows from the Knaster–Tarski fixed point theorem that there is a set A such that $A^{\Diamond\Diamond} = A$.

Question. When is it the case that $A^{\Diamond} \geqslant_e A$ (equiv., $A^{\Diamond} \equiv_e J_e(A)$)?

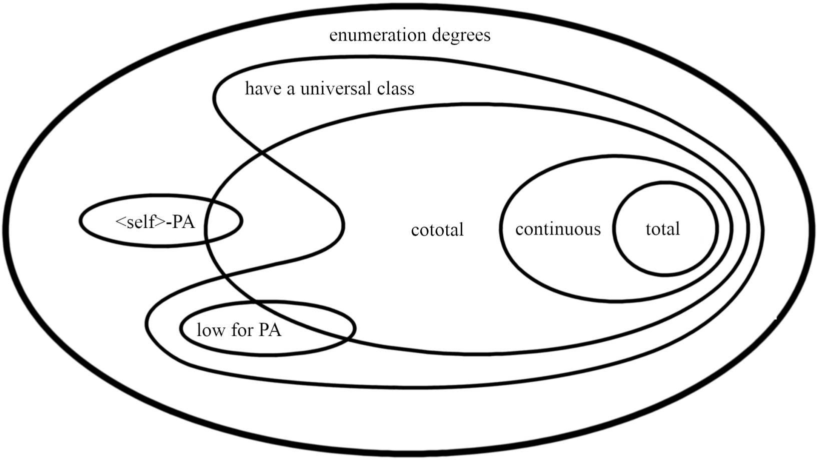
Theorem. The following are equivalent for an enumeration degree $\mathbf{a}$:

- $\mathbf{a}^{\Diamond} \geqslant_e \mathbf{a}$,
- $\mathbf{a}$ contains a *cototal* set A, i.e., $A \leqslant_e \overline{A}$,
- (McCarthy 2018) $\mathbf{a}$ contains the complement of a maximal antichain in $\omega^{<\omega}$,
- (McCarthy 2018) $\mathbf{a}$ contains the language of a minimal subshift,
- (Kihara, Ng, and Pauly) $\mathbf{a}$ is the degree of a point in a computable G_δ topological space.

We call these degrees *cototal*. They were studied extensively in (AGKLMSS 2019).

One more picture

We know where the cototal degrees fit into our picture. Note that neither cototality nor having a universal $\Pi^0_1\langle\cdot\rangle$ class implies the other.



Thank you!