Complexity of maximal objects

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$\mathcal{A} \subseteq [\omega]^\omega$ is a *MIF* if $\mathcal{A}$ is ind. and maximal with this property, i.e., for all $X \in [\omega]^\omega$ there are $\mathcal{F}, \mathcal{G} \subseteq \mathcal{A}$ finite and disjoint such that either $\sigma(\mathcal{F}, \mathcal{G}) \subseteq^* X$ or $\sigma(\mathcal{F}, \mathcal{G}) \cap X =^* \emptyset$

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Similarly for MEDs and MIFs.

Classical results

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$$g : \begin{array}{l} ([\omega]^\omega)^2 \longrightarrow \mathcal{P}\left(\omega \,\dot\cup\, 2^{<\omega}\right) \\ (x, y) \longmapsto x \,\dot\cup\, \{\chi_y \restriction n \mid n < \omega\} \end{array}$$

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Let $\mathcal{I} = g[\mathcal{F}]$. Claim that $\mathcal{I}$ is Π^1_1 MIF.

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$\sigma(z_1, \ldots, z_n; w_1, \ldots, w_\ell) \supseteq \sigma(a_1, \ldots, a_n; b_1, \ldots, b_\ell)$ is infinite, by independence of $\mathcal{I}_0$.

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<u>$\mathcal{I}$ is maximal:</u> Let $W \in [\omega \dot\cup 2^{<\omega}]^\omega$ with $W \notin \mathcal{I}$. Let $A := W \cap \omega$. By maximality of $\mathcal{I}_0$, there are distinct $a_1, \dots, a_n$ and $b_1, \dots, b_\ell \in \mathcal{I}_0$ such that $\sigma(a_1, \dots, a_n, A;\ b_1, \dots, b_\ell)$ or $\sigma(a_1, \dots, a_n;\ b_1, \dots, b_\ell, A)$ is finite, w.l.o.g. the former.
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MED families

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Note that a Borel MED necessarily has size continuum $\mathfrak{c}$.

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- Is non-existence of (projective) MADs / MIFs consistent on the basis of ZFC?

MAD and MED: the Cohen model

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In the Cohen model (over $V = L$) there is a Σ^1_2 and thus Π^1_1 MAD.

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So the witness is a union of ω_1 many closed sets of size $\mathfrak{c}$.

This is no accident b/c $\mathbf{\Sigma}^1_2$ sets are ω_1-Borel.

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This is also true for MAD and MED.

MIF: an implication

Theorem 12 (B. and Khomskii '15)

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$\mathrm{cov}(meager) > \omega_1$ implies there are no $\boldsymbol{\Sigma}^1_2$ MIFs.

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Reformulated: Adding an unbounded real to a model of set theory destroys MIF of this model.

MAD families: non-existence

Theorem 14 (Horowitz and Shelah '16)

It is consistent on the basis of ZFC that there are no (projective) MADs.

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Note. This was originally proved by Mathias under the assumption of the consistency of a Mahlo cardinal, and by Törnquist, under the assumption of the consistency of an inaccessible cardinal.

MIF: non-existence

Theorem 15 (B. and Khomskii '15)

In the Cohen model W there are no projective MIFs. In $L(\mathbb{R})^W$ there are no MIFs.

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This is based on the same combinatorics as the previous result + homogeneity of Cohen forcing.

Π^1_2 sets

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It is consistent there is a Π^1_2 MAD of size μ where $\omega_1 < \mu < \mathfrak{c}$.

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Similar for MED.