Approaching the first-order part of Ramsey's theorem for pairs and two colors

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Higher Recursion Theory and Set Theory

This talk was cancelled due to the speaker's illness.

Ramsey's theorem

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- $\mathrm{RT}^n_\infty := \forall k\ \mathrm{RT}^n_k$.
- $\mathrm{RT}^\infty_\infty := \forall n\ \mathrm{RT}^n$.

(We often omit ∞.)

We will calibrate the strength of Ramsey's theorem in the setting of reverse mathematics.

Ramsey's theorem

Proposition (RCA_0)

1. *If* $n' \leq n, k' \leq k$, *then* $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^{n'}_{k'}$.
2. $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.

Proposition (RCA_0)

For any $n \in \omega$, $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus,
$\mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 \leq \mathrm{RT}^3 \leq \mathrm{RT}^4_2 \leq \dots$

What is the strength of Ramsey's theorem?

Proposition

ACA_0 *proves* $\forall n \forall k (\mathrm{RT}^n_k \to \mathrm{RT}^{n+1}_k)$.

Theorem (Jockusch 1972)

Over RCA_0, RT^3_2 *implies* ACA_0.

Proof.

There exists a computable coloring for $[\mathbb{N}]^3$ whose homogeneous set always computes $0'$. □

Thus, for $n \geq 3$, $\mathrm{RT}^n_2 = \mathrm{RT}^n = \mathsf{ACA}_0$.

How about the strength of Ramsey's theorem for pairs?

Two important papers

- D. Seetapun and T. A. Slaman, On the strength of Ramsey's theorem. NDJFL 36 (1995).

Theorem

1. *There is an ω-model of* $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ *which avoids* $0'$*.* *Hence* $\mathsf{RCA}_0 + \mathrm{RT}^2$ *does not imply* ACA_0*.*
2. $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ *is not* Π^0_4*-conservative over* $\mathrm{I}\Sigma^0_1$*.*

For the strengthening of 2, Hirst showed that

- $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ implies $\mathrm{B}\Sigma^0_2$ ($\mathrm{B}\Sigma^0_2$ is Π^0_4),
- $\mathsf{RCA}_0 + \mathrm{RT}^2$ implies $\mathrm{B}\Sigma^0_3$.

Two important papers

- P. A. Cholak, C. G. Jockusch and T. A. Slaman, On the strength of Ramsey's theorem for pairs. JSL 66 (2001).

Theorem

1. $\mathrm{RT}^2_2 = \mathrm{COH} + \mathrm{D}^2_2$, *where*
   - COH *for any sequence of sets* $\langle R_i : i \in \mathbb{N}\rangle$*, there exists an infinite* $H \subset \mathbb{N}$ *such that* $H \subseteq^* R_i \vee H \subseteq^* R_i{}^c$ *for any* $i \in \mathbb{N}$*.*
   - D^2_2 *for any* Δ^0_2*-set* $\mathcal{A} \subseteq \mathbb{N}$*, there exists an infinite* $H \subset \mathbb{N}$ *such that* $H \subseteq \mathcal{A} \vee H \subseteq \mathcal{A}^c$
2. *(*low_2 *basis theorem) any computable* $c : [\omega]^2 \to 2$ *has a* low_2*-homogeneous set.*
3. $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2 + \mathrm{RT}^2_2$ *is* Π^1_1*-conservative over* $\mathrm{I}\Sigma^0_2$*.*
4. $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_3 + \mathrm{RT}^2_2$ *is* Π^1_1*-conservative over* $\mathrm{I}\Sigma^0_3$*.*

3 is shown by: for any $(M, S) \models \mathrm{I}\Sigma^0_2$, there exists $\bar{S} \supseteq S$ such that $(M, \bar{S}) \models \mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2 + \mathrm{RT}^2_2$.

Important questions

Question

Does $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ prove the consistency of $\mathrm{I}\Sigma^0_1$?

$\Rightarrow$ No! Actually, they are equi-consistent. (Patey/Y)

Question

Does $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ imply WKL_0?

$\Rightarrow$ No! (Liu 2012)

Question

Does $\mathrm{RCA}_0 + \mathrm{D}^2_2$ imply RT^2_2?

$\Rightarrow$ No! (Chong/Slaman/Yang 2014, Monin/Patey new!)

Question

Is $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$?

Important questions

Question

Is $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$?
What is the first-order part of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ or $\mathrm{WKL}_0 + \mathrm{RT}^2_2$?

$\Rightarrow$ Still open.

Related results

- $\mathrm{RCA}_0 + \mathrm{CAC}$ is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$ (Chong/Yang/Slaman).
- $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ does not imply $\mathrm{I}\Sigma^0_2$ (Chong/Yang/Slaman).
- $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ is Π^0_3-conservative over $\mathrm{B}\Sigma^0_2$ (Patey/Y).
- $\mathrm{WKL}_0 + \mathrm{RT}^2$ is Π^1_1-conservative over $\mathrm{B}\Sigma^0_3$ (Slaman/Y).

$\vdots$

Standard approach: ω-extension

Question

Is $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$?

To show this conservation, it is enough to show the following:

- for any (countable) $(M, S) \models \mathrm{B}\Sigma^0_2$, and $A \in S$, there exists $\bar{S} \subseteq \mathcal{P}(M)$ such that $A \in \bar{S}$ and $(M, \bar{S}) \models \mathsf{RCA}_0 + \mathrm{RT}^2_2$ (or $(M, \bar{S}) \models \mathsf{RCA}_0 + \mathrm{D}^2_2$).

Theorem (Cholak/Jockusch/Slaman)

For any countable $(M, S) \models \mathrm{I}\Sigma^0_2$, there exists $S \subseteq \bar{S} \subseteq \mathcal{P}(M)$ such that $(M, \bar{S}) \models \mathsf{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$.

- Indeed, for any $c : [M]^2 \to 2$ in S, one may construct an infinite homogeneous set $H \subseteq M$ by (a variant of) Mathias forcing so that $(M, S \cup \{H\}) \models \mathrm{I}\Sigma^0_2$.

Thus, we only need to consider the case of $(M, S) \models \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$.

Can we preserve $\mathrm{B}\Sigma^0_2$?

Let $(M, S) \models \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$. Try to find an ω-extension $(M, \bar{S}) \models \mathrm{D}^2_2$

Question

For a Δ^0_2 set $\mathcal{A}$ in (M, S), can we construct an infinite set $H \subseteq \mathcal{A}$ or $H \subseteq \mathcal{A}^c$ such that $(M, S \cup \{H\}) \models \mathrm{B}\Sigma^0_2$?

This is not easy.

- by Mathias forcing? $\Rightarrow$ to force $\mathrm{B}\Sigma^0_2$, Σ^0_2-induction is needed (and essential?)
- constructing definable homogeneous set? $\Rightarrow$ to preserve $\mathrm{B}\Sigma^0_2$, we need a low solution, but that is usually impossible(?)

One cannot construct a low solution in general.

Theorem (Downey/Hirschfeldt/Lempp/Solomon)

If $(M, S) \models \mathrm{I}\Sigma^0_2$, there exists a Δ^0_2-set $\mathcal{A}$ in (M, S) such that there is no infinite low set $H \subseteq M$ such that $H \subseteq \mathcal{A}$ or $H \subseteq \mathcal{A}^c$.

Still, there is some hope.

Theorem (Chong/Slaman/Yang)

Let $(M, S) \models \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2 + \mathrm{BME}_{<\infty}$, and has strong enough saturation plus suitable condition. Then, for any Δ^0_2-set $\mathcal{A}$ in (M, S), there exists an infinite low set $H \subseteq M$ such that $H \subseteq \mathcal{A}$ or $H \subseteq \mathcal{A}^c$.

- What is $\mathrm{BME}_{<\infty}$?

  BME_n: n-th iterated bounded monotone enumeration.

  (We omit the original definition, but it is equivalent to the well-foundedness property.)

BME_n is characterized as follows.

Theorem (Kreuzer/Y)

Let $n \geq 1$. Over RCA_0, BME_n is equivalent to $\mathrm{WF}(\omega_n)$.
(Here, $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$.)

- $\mathrm{BME}_{<\infty}$ or at least BME_1 seems to be essential in the construction by Chong/Slaman/Yang.
- Then, $\mathrm{BME}_1 (= \mathrm{WF}(\omega^\omega))$ is provable from $\mathsf{WKL}_0 + \mathrm{RT}^2_2$?

  $\Rightarrow$ No!

  Indeed, $\mathrm{WF}(\omega^\omega)$ implies the consistency of $\mathrm{I}\Sigma^0_1$, but $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ is equi-consistent with $\mathrm{I}\Sigma^0_1$.

Maybe some smaller ordinal?

- RCA_0 proves $\mathrm{WF}(\omega^k)$ for any standard number k, but this seems not enough for CSY construction.

Seeking for a counterexample?

Is there some good candidate for a counterexample? If φ is a counterexample to the Π^1_1-conservation, it should satisfies the following:

- φ is provable from $\mathrm{I}\Sigma^0_2$ (since the first-order part of RT^2_2 is weaker than $\mathrm{I}\Sigma^0_2$),
- φ is not provable from $\mathrm{B}\Sigma^0_2$,
- $\mathsf{WKL}_0 + \mathrm{B}\Sigma^0_2 + \varphi$ is Π^0_3-conservative over RCA_0.

Finite combinatorial principles like PH^2_k are often related with $\mathrm{WF}(\alpha)$. Is there some reasonable candidate of this type?

For example, $\forall n(\mathrm{WF}(\omega^n) \to \mathrm{WF}(\omega^{2^n}))$ satisfies the above three conditions.
(This statement is not Π^1_1, but we can find a Π^1_1-fragment of this statement which still satisfies the conditions.)

Seeking for a counterexample?

Is $\forall n(\mathrm{WF}(\omega^n) \to \mathrm{WF}(\omega^{2^n}))$ provable within $\mathsf{WKL}_0 + \mathrm{RT}^2_2$?

$\Rightarrow$ No!

Theorem (Kołodziejczyk/Y)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is conservative over* RCA_0 *with respect to sentences of the form:*

$$\forall \alpha < \omega^\omega \, (\mathrm{WF}(\alpha) \to \varphi(\alpha))$$

where φ *is* $\tilde{\Pi}^0_3$.

Corollary

For any primitive recursive function $p \colon \omega^\omega \to \omega^\omega$ *(defined on codes of ordinals), if* $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *proves*

$$\forall \alpha < \omega^\omega \, (\mathrm{WF}(\alpha) \to \mathrm{WF}(p(\alpha))),$$

then RCA_0 *proves the same statement.*

Theorem (Y)

$\mathrm{WKL}_0^* + \mathrm{RT}_2^2$ *is* $\tilde{\Pi}_3^0$*-conservative over* RCA_0^*.

Could it be Π_1^1-conservative?

Remark

- $\mathrm{WKL}_0^* + \mathrm{RT}_2^2$ implies $\mathrm{I}\Sigma_1^0 \to \mathrm{B}\Sigma_2^0$, or more precisely, $\forall X\, \mathrm{I}\Sigma_1^X \to \forall X\, \mathrm{B}\Sigma_2^X$. However, this statement is not Π_1^1. On the other hand, $\mathrm{WKL}_0^* + \mathrm{RT}_2^2$ does not imply $\forall X(\mathrm{I}\Sigma_1^X \to \mathrm{B}\Sigma_2^X)$.
- $\mathrm{I}\Sigma_1^0$ is equivalent to "any infinite set contains x many elements for any $x \in \mathbb{N}$".

In case $\mathrm{I}\Sigma_1^0$ is absent, $\mathrm{RCA}_0^* + \mathrm{RT}_2^2$ still implies the following (by the lower bound of finite Ramsey's theorem for pairs):

(†) if any infinite set contains x many elements, then any infinite set contains 2^x many elements.

The case over RCA_0^*

Once induction is absent, "cuts" play central roles for arithmetical consequences. Let

$$\mathcal{I}_n^X := \bigcap\{I : I \text{ is a } \Sigma_n^X\text{-definable cut}\}.$$

The following Wong's theorem is very useful to analyze cuts in second-order arithmetic.

Theorem (Wong)

Let $n \geq 1$. The following is provable within $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0$.

$$\text{For any } X, Y,\ \mathcal{I}_n^X = \mathbb{N} \vee \mathcal{I}_n^Y = \mathbb{N} \vee \mathcal{I}_1^X = \mathcal{I}_1^Y.$$

(Here $\mathbb{N} = \{x : x = x\}$ is the set of all numbers.)

Over RCA_0^*, we see $\mathcal{I}_1^X = \bigcap\{\,|Y| : Y \leq_T X \text{ and } Y \text{ is unbounded}\}$.

The case over RCA_0^*

Over RCA_0^*, we see $\mathcal{I}_1^X = \bigcap\{\,|Y| : Y \leq_T X \text{ and } Y \text{ is unbounded}\}$.

Then, (†) is rephrased as follows:

(†) if any infinite set contains x many elements, then any infinite set contains 2^x many elements.

$\Leftrightarrow$ for any n, if $(\forall X\, n \in \mathcal{I}_1^X)$ then $(\forall X\, 2^n \in \mathcal{I}_1^X)$.

By Wong's theorem, this is equivalent to Π^1_1-statement (†′).

(†′) for any X and for any n, if $n \in \mathcal{I}_1^X$ then $2^n \in \mathcal{I}_1^X$.

On the other hand, (†) is not provable from RCA_0^* since

$\mathrm{RCA}_0^* + (\dagger)$ has super-exponential speed-up over RCA_0^* to prove "$\exists m\, m = \mathit{supexp}(\mathit{supexp}(\mathit{supexp}(n)))$" for $n = 0, 1, 2, \dots$

(model theoretic separation is also available)

Corollary

$\mathrm{RCA}_0^* + \mathrm{RT}^2_2$ *is not* Π^1_1*-conservative over* RCA_0^*.

Back on topic

Can we use the same argument for $\mathrm{RCA}_0 + \mathrm{RT}^2_2$?

Over RCA_0, we see $\mathcal{I}^X_2 = \bigcap\{\,|\mathcal{A}| : \mathcal{A}$ is Π^X_1-definable and unbounded$\}$.

Then, we consider the following $(\dagger\dagger)$:

$(\dagger\dagger)$ for any n, if $(\forall X\, n \in \mathcal{I}^X_2)$ then $(\forall X\, 2^n \in \mathcal{I}^X_2)$.

This $(\dagger\dagger)$ may be a good candidate for a counterexample of Π^1_1-conservation since

- $(\dagger\dagger)$ is Π^1_1 by Wong's theorem,
- $\mathrm{I}\Sigma^0_2$ proves $(\dagger\dagger)$,
- $\mathrm{WKL}_0 + (\dagger\dagger)$ is $\tilde{\Pi}^0_3$-conservative over RCA_0,
- $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$ does not imply $(\dagger\dagger)$.

Counterexample?

(††) for any n, if $(\forall X\, n \in \mathcal{I}_2^X)$ then $(\forall X\, 2^n \in \mathcal{I}_2^X)$.

(††) is provable from the following version of RT^2_2 (when $\mathrm{I}\Sigma^0_1$ is absent).

Proposition (RT^2_2 on a Π^0_1-unbounded set)

Let $\mathcal{A}$ be a Π^0_1-definable unbounded set such that $|\mathcal{A}| < a$ for some $a \in \mathbb{N}$. Let $c : [[0, a]]^2 \to 2$ be a finite coloring. We may consider c as a coloring on $[\mathcal{A}]^2$ by identifying i-th smallest element of $\mathcal{A}$ with $i \in [0, a]$. Then, there exists a Σ^0_2-definable subset (possibly with a new parameter) $\mathcal{H} \subseteq \mathcal{A}$ which is unbounded and homogeneous for c.

Question

Is this provable within $\mathrm{WKL}_0 + \mathrm{RT}^2_2$?

If the answer is yes, then $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is not Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$.

Thank you!

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