BQO-Wadge degrees on nonseparable ultrametric spaces and computability on uncountable cardinals

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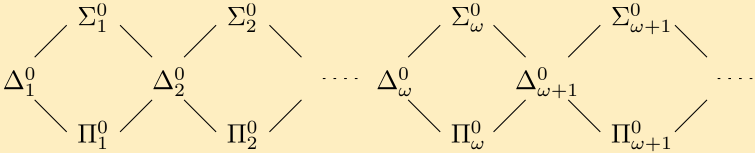
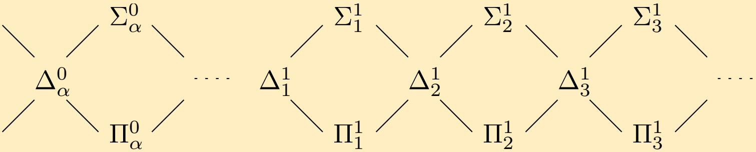
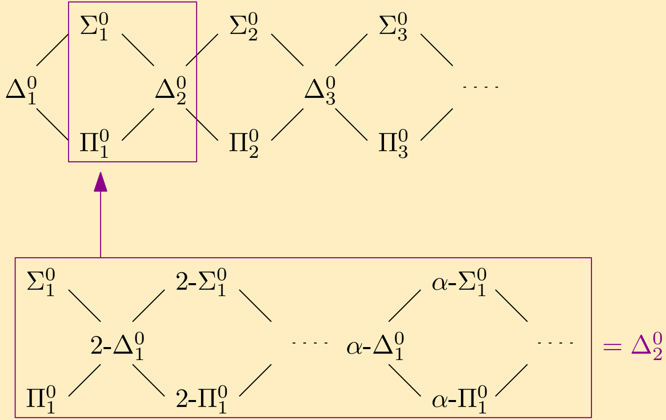
Borel hierarchy



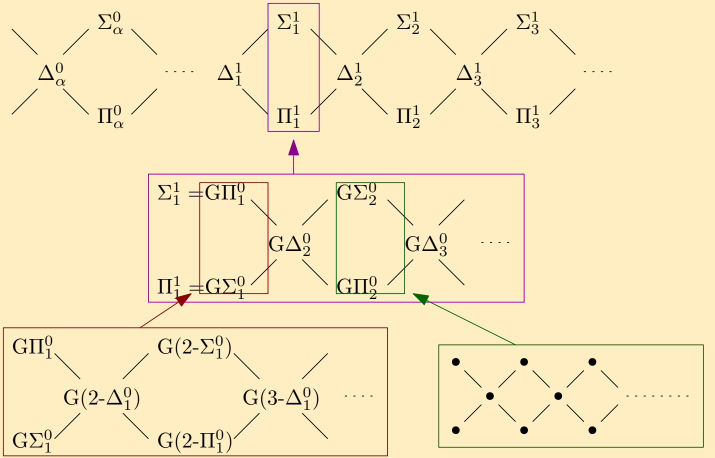
Projective hierarchy



Hausdorff-Kuratowski difference hierarchy



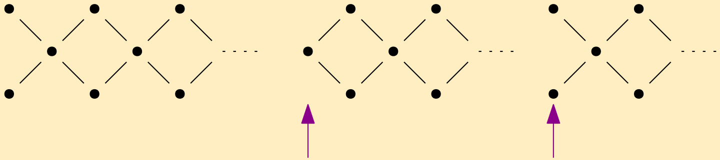
Hierarchies inside $\mathbf{\Delta^1_2}$



The C-hierarchy (Selivanovskii, Nikodym)

The R-hierarchy (Kolmogorov)

Wadge hierarchy (1970s)

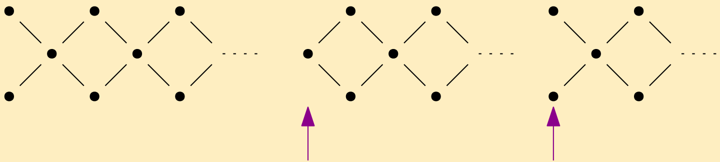


- Ultimate measure for topological complexity:
  For subsets $A, B \subseteq X$ of a topological space X

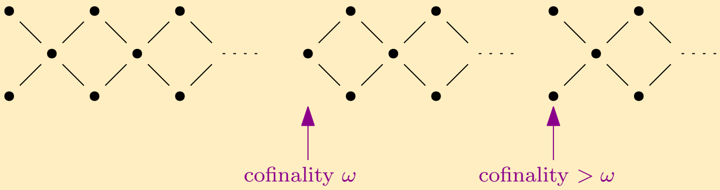
$$A \leq_W B \iff A = \theta^{-1}[B] \text{ for some continuous } \theta\colon X \to X$$
$$\iff \chi_A = \chi_B \circ \theta \text{ for some continuous } \theta\colon X \to X$$

Theorem (Wadge + Martin-Monk)

(AD) The subsets of ω^ω are semi-well-ordered by $\leq_W$.



- The structure of Wadge degrees of subsets of Baire space (or functions $f : \omega^\omega \to \mathbf{2}$) is extremely simple and beautiful.
- However, the structure is too simple to analyze why the structure is so simple.



- The structure of Wadge degrees of subsets of Baire space (or functions $f : \omega^\omega \to \mathbf{2}$) is extremely simple and beautiful.
- However, the structure is too simple to analyze why the structure is so simple.
- Are there any domains other than ω^ω and codomains other than $\mathbf{2}$ having an understandable Wadge degree structure?

Let S be a topological space, and Q be a preorder.

- $f : S \to Q$ is Wadge reducible to $g : S \to Q$ (written $f \leq_W g$) if there is a continuous $\theta : S \to S$ s.t. $f(x) \leq_Q g \circ \theta(x)$ for all $x \in S$.

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Homomorphic Quasi-Ordering (Hertling, Selivanov, and others)

$\mathcal{A} = (A, \leq_A, c_A)$, $\mathcal{B} = (B, \leq_B, c_A)$: Q-labeled preorders (i.e., $c_X : X \to Q$)

- A morphism $h : \mathcal{A} \to \mathcal{B}$ is a function $h : A \to B$ s.t.
  - $x \leq_A y \implies h(x) \leq_B h(y)$. (order preserving)
  - $c_A(x) \leq_Q c_B(h(x))$. (label increasing)
- Write $\mathcal{A} \leq_h \mathcal{B}$ if there is a morphism $h : \mathcal{A} \to \mathcal{B}$.
- The quasi-ordering $\leq_h$ is called a homomorphic quasi-order.

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Theorem (Selivanov 2007)

Let k be a finite discrete order. Then the following are isomorphic:

- The Wadge degrees of k-valued $\underset{\sim}{\Delta}^0_2$ functions on ω^ω
- The quotient homomorphic quasiorder on countable well-founded k-labeled forests.

- ${}^{\sqcup}S$: The set of all S-labeled countable discrete orders.
- **Tree**(S): The set of all S-labeled well-founded countable trees.
- ${}^{\sqcup}$**Tree**(S): The set of all S-labeled well-founded countable forests.

Theorem (K. and Montalbán, TAMS 2019)

Let Q be a BQO (better-quasi-order).

- The Wadge degrees of $\Delta^0_1(\omega^\omega \to Q) \simeq {}^{\sqcup}Q$.
- The Wadge degrees of $\Delta^0_2(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Wadge degrees of $\Delta^0_3(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Wadge degrees of $\Delta^0_4(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Wadge degrees of $\Delta^0_5(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

In general, for $\alpha < \omega_1$, the Wadge degrees of $\Delta^0_{1+\alpha}(\omega^\omega \to Q)$ is characterized as a generalized version of homomorphic quasi-order on ${}^{\sqcup}\mathbf{Tree}^{\alpha}(Q)$.

WQO/BQO theory

Q is *well-quasi-ordered* (WQO) $\iff$ Q has no infinite decreasing sequence nor infinite antichain.

Higman (1952), Kruskal (1960)

- Q is WQO $\implies$ The finite subsets $\mathcal{P}_{<\omega}(Q)$ are WQO:
- Q is WQO $\implies$ The finite sequences $Q^{<\omega}$ are WQO:
- Q is WQO $\implies$ The finite trees $\mathbf{Tree}_{<\omega}(Q)$ are WQO:

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  - For $A, B \subseteq Q$, $A \leq B \iff (\forall p \in A)(\exists q \in B)\ p \leq_Q q$.
  - This is equivalent to the homomorphic quasi-ordering on the Q-labeled finite discrete orders.
- Q is WQO $\implies$ The finite sequences $Q^{<\omega}$ are WQO:
  - For $\alpha, \beta \in Q^{<\omega}$, $\alpha \leq \beta \iff \exists$ order-preserving injection $\theta : \mathbf{dom}(\alpha) \to \mathbf{dom}(\beta)$ s.t. $\alpha(n) \leq_Q \beta \circ \theta(n)$.
  - This is equivalent to the injective homomorphic quasi-ordering on the Q-labeled finite well-orders.
- Q is WQO $\implies$ The finite trees $\mathbf{Tree}_{<\omega}(Q)$ are WQO:
  - under inf-preserving injective homomorphic quasi-ordering.

Q is *well-quasi-ordered* (WQO) $\iff$

For every $h : \omega \to Q$ there are $i < j$ such that $h(i) \leq_Q h(j)$.

Q is *better-quasi-ordered* (BQO) $\iff$ For every Borel function $h : [\omega]^\omega \to Q$ there is $(\alpha_n)_n \in [\omega]^\omega$ such that $h((\alpha_n)_n) \leq_Q h((\alpha_{n+1})_n)$.

Nash-Williams (1965,1968), Laver (1968)

- Q is BQO $\implies$ The subsets $\mathcal{P}(Q)$ are BQO:
- Q is BQO $\implies$ The ordinal-length sequences $Q^{<\mathrm{Ord}}$ are BQO:
- Q is BQO $\implies$ The trees $\mathcal{T}(Q)$ of arbitrary cardinality, but of height $\leq \omega$, are BQO:

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- Q is BQO $\implies$ The trees $\mathcal{T}(Q)$ of arbitrary cardinality, but of height $\leq \omega$, are BQO:
  - under inf-preserving injective homomorphic quasi-ordering.

- (van Engelen-Miller-Steel 1987; Block 2014; K.-Montalbán)
  Q is BQO $\Longrightarrow$ $\mathbf{Borel}(\omega^\omega, Q)$ is BQO under $\leq_W$.
- (AD$^+$) Q is BQO $\Longrightarrow$ $\mathbf{Func}(\omega^\omega, Q)$ is BQO under $\leq_W$.

Hence:

- If Q is BQO, then $\underset{\sim}{\mathbf{\Delta}}^0_1(\omega^\omega, Q)$ is BQO under $\leq_W$,
  so the countable subsets $\mathcal{P}_{\leq\omega}(Q)$ are BQO.
- If Q is BQO, then $\underset{\sim}{\mathbf{\Delta}}^0_2(\omega^\omega, Q)$ is BQO under $\leq_W$,
  so the countable well-founded forests ${}^{\sqcup}\mathbf{Tree}_{\leq\omega}(Q)$ are BQO.

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It is natural to expect that

- If Q is BQO, then $\utilde{\mathbf{\Delta}}^0_1(\kappa^\omega, Q)$ is BQO under $\leq_W$,
  so the subsets $\mathcal{P}_{\leq\kappa}(Q)$ of cardinality $\leq$ κ are BQO.
- If Q is BQO, then $\utilde{\mathbf{\Delta}}^0_2(\kappa^\omega, Q)$ is BQO under $\leq_W$,
  so the well-founded forests ${}^{\sqcup}\mathbf{Tree}_{\leq\kappa}(Q)$ of cardinality $\leq$ κ are BQO.

BQO-Wadge degrees on ω^ω

- ${}^{\sqcup}S$: The set of all S-labeled countable discrete orders.
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In general, for $\alpha < \omega_1$, the Wadge degrees of $\underset{\sim}{\Delta}^0_{1+\alpha}(\omega^\omega \to Q)$ is characterized as a generalized version of homomorphic quasi-order on ${}^{\sqcup}\mathbf{Tree}^\alpha(Q)$.

Hereafter, we will describe a tree and a forest as a term in the following language:

Language $\mathcal{L}(Q)$ for nested Q-labeled trees and forests

1. Constant symbols q (for $q \in Q$).
2. A 2-ary function symbol $\rightarrow$ (concatenation).
3. An ω-ary function symbol $\sqcup$ (disjoint sum).
4. A unary function symbol $\langle \cdot \rangle$ (labeling).

Example of $\mathcal{L}(2)$-terms

1. The term $0^{\rightarrow}1$ represents open sets (c.e. sets).
2. The term $1^{\rightarrow}0$ represents closed sets (co-c.e. sets).
3. The term $0 \sqcup 1$ represents clopen sets (computable sets).
4. The term $0^{\rightarrow}1^{\rightarrow}0$ represents differences of two open sets (d-c.e. sets).
5. The term $\langle 0^{\rightarrow}1 \rangle$ represents F_σ sets (Σ^0_2 sets).

Definition (the class Σ_T; K.-Montalbán)

For a $\mathcal{L}(Q)$-term T, define the class Σ_T of functions: $\omega^\omega \to Q$ as follows

1. Σ_q consists only of the constant function $x \mapsto q$.
2. $f \in \Sigma_{\sqcup_i S_i} \iff \exists$ clopen partition $(C_i)_{i\in\omega}$ of ω^ω s.t. $f \upharpoonright C_i$ is in Σ_{S_i}.
3. $f \in \Sigma_{S\to T} \iff$ there is an open set $U \subseteq \omega^\omega$ such that $f \upharpoonright U$ is in Σ_T and $f \upharpoonright (\omega^\omega \setminus U)$ is in Σ_S.
4. $f \in \Sigma_{\langle T\rangle} \iff f = g \circ h$ for some $g \in \Sigma_T$ and Σ^0_2-measurable h.

Note: For the 3rd condition, without affecting its Wadge degree, a function with closed or open domain can be modified to a total function.

1. $\Sigma_{0\to1} = \mathbf{\Sigma}^0_1$, $\Sigma_{1\to0} = \mathbf{\Pi}^0_1$, and $\Sigma_{0\sqcup1} = \mathbf{\Delta}^0_1$.
2. $\Sigma_{0\to1\to0}$ = differences of $\mathbf{\Sigma}^0_1$ sets.
3. $\Sigma_{\langle 0\to1\rangle} = \mathbf{\Sigma}^0_2$, and $\Sigma_{\langle 1\to0\rangle} = \mathbf{\Pi}^0_2$.
4. No term corresponds to $\mathbf{\Delta}^0_2$ (this reflects the fact that there is no $\mathbf{\Delta}^0_2$-complete set; $\mathbf{\Delta}^0_2$ is divided into unbounded ω_1-many Wadge degrees).

- ${}^{\sqcup}\mathbf{Tree}^{\omega}(Q)$: the set of all terms in the previous language.
- $\mathbf{Tree}^{\omega}(Q)$: the set of all terms whose outmost symbol is not $\sqcup$.
- Incorrect Idea: $\mathbf{Tree}(\mathbf{Tree}^{\omega}(Q))$ may describe the rank $\omega + 1$.

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- ${}^{\sqcup}\mathbf{Tree}^{\omega}(\mathcal{Q})$: the set of all terms in the previous language.
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- Correct Idea: $\mathbf{Tree}^{\omega}(\mathbf{Tree}(\mathcal{Q}))$ describes the rank $\omega + 1$.
- To deal with this, we have to distinguish two labeling functions for inner trees $\mathbf{Tree}(\mathcal{Q})$ and the outer tree $\mathbf{Tree}^{\omega}$.

- Given an ordering $\mathcal{P}$, define the new ordering $\langle\mathcal{P}\rangle^{\omega}$ as follows:
  - $\langle\mathcal{P}\rangle^{\omega} = \{\langle p\rangle^{\omega} : p \in \mathcal{P}\}$.
  - $\langle p\rangle^{\omega} \leq \langle q\rangle^{\omega} \iff p \leq_{\mathcal{P}} q$.
- Obviously $\langle\mathcal{P}\rangle^{\omega}$ is isomorphic to $\mathcal{P}$.
- To avoid a notational confusion, instead of $\mathbf{Tree}^{\omega}(\mathbf{Tree}(\mathcal{Q}))$, we will consider $\mathbf{Tree}^{\omega+1}(\mathcal{Q}) := \mathbf{Tree}^{\omega}(\langle\mathbf{Tree}(\mathcal{Q})\rangle^{\omega})$.
- Continue, e.g. $\mathbf{Tree}^{\omega\cdot 2+6}(\mathcal{Q}) := \mathbf{Tree}^{\omega}(\langle\mathbf{Tree}^{\omega}(\langle\mathbf{Tree}^{6}(\mathcal{Q})\rangle^{\omega})\rangle^{\omega})$, which is isomorphic to $\mathbf{Tree}^{\omega}(\mathbf{Tree}^{\omega}(\mathbf{Tree}^{6}(\mathcal{Q})))$.

Language $\mathcal{L}_\alpha$ for nested Q-labeled trees and forests

1. Constant symbols q (for $q \in Q$).
2. A 2-ary function symbol $\to$ (concatenation).
3. An ω-ary function symbol $\sqcup$ (disjoint sum).
4. A unary function symbol $\langle \cdot \rangle^{\omega^\beta}$ for every $\beta < \alpha$ (ω^β-labeling).

- $^{\sqcup}\mathbf{Tree}^{\omega^\alpha}(Q)$: The set of all $\mathcal{L}_\alpha$-terms.
- $\mathbf{Tree}^{\omega^\alpha}(Q)$: The set of all $\mathcal{L}_\alpha$-terms whose outmost symbol is not $\sqcup$.
- Every $\xi < \omega_1$ is uniquely decomposed as $\xi = \omega^\alpha + \beta$.
  Then, inductively define $\mathbf{Tree}^{\xi}(Q) := \mathbf{Tree}^{\omega^\alpha}(\langle \mathbf{Tree}^{\beta}(Q) \rangle^{\omega^\alpha})$.
- $\mathbf{Tree}^{\omega_1}(Q) = \bigcup_{\xi<\omega_1} \mathbf{Tree}^{\xi}(Q)$.
- $^{\sqcup}\mathbf{Tree}^{\omega_1}(Q) = \bigcup_{\xi<\omega_1} {}^{\sqcup}\mathbf{Tree}^{\xi}(Q)$.

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4. $f \in \Sigma_{\langle T\rangle^{\omega^\alpha}} \iff f = g \circ h$ for some $g \in \Sigma_T$ and $h \in \Sigma^0_{1+\omega^\alpha}$.

We define a quasi-order $\trianglelefteq$ on terms, which is shown to be isomorphic to the Wadge degrees of finite Borel rank.

Definition of $\trianglelefteq$ (K.-Montalbán)

We inductively define a quasi-order $\trianglelefteq$ on terms as follows:

$$p \trianglelefteq q \iff p \leq_Q q,$$

$$\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff \begin{cases} U \trianglelefteq V & \text{if } \alpha = \beta, \\ \langle U\rangle^{\omega^\alpha} \trianglelefteq V & \text{if } \alpha > \beta, \\ U \trianglelefteq \langle V\rangle^{\omega^\alpha} & \text{if } \alpha < \beta. \end{cases}$$

and if S and T are of the form $\langle U\rangle^{\omega^\alpha} \to \bigsqcup_i S_i$ and $\langle V\rangle^{\omega^\beta} \to \bigsqcup_j T_j$, then

$$S \trianglelefteq T \iff \begin{cases} (\forall i)\ S_i \trianglelefteq T & \text{if } \langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta}, \\ (\exists j)\ S \trianglelefteq T_j & \text{if } \langle U\rangle^{\omega^\alpha} \ntrianglelefteq \langle V\rangle^{\omega^\beta}. \end{cases}$$

A function $f : \omega^\omega \to Q$ is Σ_T-complete if $f \in \Sigma_T$, and every Σ_T-function $g : \omega^\omega \to Q$ is Wadge reducible to f.

Lemma

For any $T \in \mathbf{Tree}^{\omega_1}(Q)$, there is a Σ_T-complete function.

Ω_T: any Σ_T-complete function.

Main Lemma 1 ($T \mapsto \Omega_T$ is an embedding)

$S \trianglelefteq T \iff \Omega_S \leq_W \Omega_T$.

Main Lemma 2 ($T \mapsto \Omega_T$ is a surjection)

Every $\underset{\sim}{\Delta}^0_{1+\xi}$-measurable function $f : \omega^\omega \to Q$ is Wadge equivalent to Ω_T for some $T \in {}^{\sqcup}\mathbf{Tree}^{\xi}(Q)$.

Definition of Σ_T-complete functions

- $\Omega_q : x \mapsto q$, a constant function.
- $\Omega_{\sqcup_i T_i}(n^\frown x) = \Omega_{T_n}(x)$.
- $\Omega_{\langle S\rangle \to F}$ is a function, which given $x \in \omega^\omega$,
  - until seeing the "mind-change" symbol in x, simulate $\Omega_{\langle S\rangle}$.
  - If we see the "mind-change" symbol in x, then erase all previous outputs, and now start to simulate Ω_F.
- $\Omega_{\langle T\rangle^{\omega^\alpha}} = \Omega_T \circ \mathrm{Lim}_{\omega^\alpha}$

where $\mathrm{Lim}_{\omega^\alpha}$ is a complete partial $\utilde{\Sigma}^0_{1+\omega^\alpha}$-measurable function.

Lemma

Ω_T is Σ_T-complete for any $T \in \mathrm{Tree}^{\omega_1}(Q)$.

Proof Techniques:
(Uniform, Transfinite) Jump Inversion for Turing degrees

The key notion in our proofs is the Turing jump operator.

The Turing jump $TJ : \omega^\omega \to \omega^\omega$ has the following properties:

- The Turing jump TJ is an right-complete Σ^0_2-computable function: For all Σ^0_2-comp. $g : \omega^\omega \to \omega^\omega$, there is a comp. θ s.t. $g = \theta \circ TJ$.
- The Turing jump TJ is an injection whose image is Π^0_2.
- The left inverse TJ^{-1} is a computable map with a Π^0_2 domain. (For computability, note that X' computes X in a uniform manner!)

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- The Turing jump $\boldsymbol{TJ}$ is an injection whose image is $\boldsymbol{\Pi}^0_2$.
- The left inverse $\boldsymbol{TJ}^{-1}$ is a computable map with a $\boldsymbol{\Pi}^0_2$ domain. (For computability, note that X' computes X in a uniform manner!)

Marcone-Montalbán (2011), in "*The Veblen functions for computability theorists*", introduced the notion of the *Turing jump via true stages*.

The MM Turing jump $\mathcal{J}$ is better than the traditional Turing jump $\boldsymbol{TJ}$:

- The MM Turing jump $\mathcal{J}$ is an right-complete $\boldsymbol{\Sigma}^0_2$-comp. function.
- The MM Turing jump $\mathcal{J}$ is an injection whose image is closed.
- The left inverse $\mathcal{J}^{-1}$ is a computable map with a closed domain.

$\mathcal{J}^Z$: The MM Turing jump relative to Z.

Since the domain $\mathcal{J}^Z$ is closed, we can think of it as a total function.

The Jump Inversion Operator $\nrightarrow$

For a function $f : \omega^\omega \to Q$, define $f^{\nrightarrow Z} = f \circ (\mathcal{J}^Z)^{-1}$.

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The Jump Inversion Operator $\nrightarrow$

For a function $f : \omega^\omega \to Q$, define $f^{\nrightarrow Z} = f \circ (\mathcal{J}^Z)^{-1}$.

- $X \leq_T Y \implies f^{\nrightarrow X} \geq_W f^{\nrightarrow Y}$.
- Since the Turing degrees form a upper semilattice, and Wadge degrees of Q-valued func. are well-founded, there must exist Z s.t.
$$\mathbf{deg}_W f^{\nrightarrow Z} = \mathbf{min}_X \mathbf{deg}_W f^{\nrightarrow X}.$$
- Define $f^{\nrightarrow} = f^{\nrightarrow Z}$ for such a Z.

Key Lemma (Jump Inversion)

$(h \circ \mathbf{Lim})^{\not\nearrow} \equiv_W h$. In particular, $\Omega^{\not\nearrow}_{\langle T \rangle} \equiv_W \Omega_T$.

- Our jump inversion theorem follows from (the uniform version of) the Friedberg jump inversion theorem.
- Our above result seems related to Brattka's work on "*A Galois connection between Turing jumps and limits*."

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Definition (Louveau and Saint-Raymond 1990)

A function $f : \omega^\omega \to Q$ is *self-dual* if

$$\exists \text{ continuous } \theta \text{ s.t. for all } x \in \omega^\omega, f \circ \theta(x) \not\leq_Q f(x).$$

Key Lemma (Jump-Inversion-Cancellation)

Assume $f^{\star}$ or $g^{\star}$ is non-self-dual. Then $f^{\star} \leq_W g^{\star} \implies f \leq_W g$.

- This follows from the (Borel) determinacy.
- The argument is almost the same as Duparc (2001), "*Wadge hierarchy and Veblen hierarchy Part I: Borel sets of finite rank*."

$\mathcal{J}^{\omega^\alpha, Z}$: The ω^α-th MM Turing jump operator relative to Z.
The left-inverse $(\mathcal{J}^{\omega^\alpha, Z})^{-1}$ is also continuous with closed domain.
Define $f^{\nrightarrow\omega^\alpha}$ as one realizing the least Wadge deg. among $f \circ (\mathcal{J}^{\omega^\alpha, Z})^{-1}$.

Transfinite Jump Inversion Theorem

$(h \circ \mathbf{Lim}_{\omega^\alpha})^{\nrightarrow\omega^\alpha} \equiv_W h$. In particular, $\Omega^{\nrightarrow\omega^\alpha}_{\langle T\rangle^{\omega^\alpha}} \equiv_W \Omega_T$.

- This follows from (the uniform version of) the transfinite jump inversion theorem.

Transfinite Jump-Inversion-Cancellation

Assume $f^{\nrightarrow\omega^\alpha}$ or $g^{\nrightarrow\omega^\alpha}$ is non-self-dual. Then $f^{\nrightarrow\omega^\alpha} \leq_W g^{\nrightarrow\omega^\alpha} \implies f \leq_W g$.

- This again follows from the (Borel) determinacy.

A function $f : \omega^\omega \to Q$ is α-*stable* if

- f is initializable, i.e., $(\exists g \equiv_W f)(\forall \sigma \in \omega^{<\omega})\ g \upharpoonright [\sigma] \equiv_W g$.
- $f^{\not\to \omega^\beta} \equiv_W f$ for any $\beta < \alpha$.

Key Technical Lemma on α-Stability

1. $\Omega_{\langle T \rangle^{\omega^\alpha}}$ is α-stable.
2. If f is α-stable, then $f^{\not\to \omega^\alpha}$ is non-self-dual.

- (1) again uses the (transfinite) jump inversion theorem.
- (2) for $\alpha = 0$ just follows from a simple finite extension method as in Duparc (2001).
- (2) for $\alpha > 0$ involves some inverse limit construction.

BQO-Wadge degrees on κ^ω

Most basic results on $\underset{\sim}{\Delta}^1_1$ BQO-Wadge degrees can be generalized to κ^ω

- Martin (1990): A quasi-Borel set is a set constructed from open sets by taking complement, countable union, and open separated union.
- In complete ultrametric spaces: Borel $\subsetneq$ quasi-Borel = $\underset{\sim}{\Delta}^1_1$.
- Martin (1990): Every quasi-Borel set is determined.

Lemma

1. Q is BQO $\implies$ $\underset{\sim}{\Delta}^1_1(\kappa^\omega, Q)$ is BQO under $\leq_W$.
2. For a $\underset{\sim}{\Delta}^1_1$ function $g : \kappa^\omega \to Q$,

$$g \text{ is Lipschitz self-dual} \iff g \text{ is Wadge self-dual.}$$

3. For a $\underset{\sim}{\Delta}^1_1$ function $g : \kappa^\omega \to Q$,

$$g \text{ is self-dual} \iff g \text{ is } \kappa\text{-join-reducible}$$

where g is κ-join-reducible if $g \equiv_W \bigoplus_{\lambda<\kappa} g_\lambda$ for some $g_\lambda <_W g$.

(Open Question) How about non-$\underset{\sim}{\Delta}^1_1$ functions? (Note: $\mathbf{AD}_{\aleph_1}$ is false.)

- All “descriptive-set-theoretic” lemmas in K.-Montalbán’s proof can be generalized to κ^ω.
- We need to generalize “computability-theoretic” lemmas to κ^ω.
  - The Turing jump, the jump inversion theorem, etc. in κ^ω.
  - The usual admissible recursion theories (e.g., α-recursion theory) are not suitable here, since they destroy topological structures.
- Moreover, there is some difficulty for defining the jump:
  - It may seem natural to define the κ-jump of x as the Σ_1-truth in $L_\kappa[x]$ with parameters; however this is an element in 2^κ, but not in κ^ω.
  - Most reasonable candidates for the κ-jump seem elements in 2^κ or κ^κ, except for the parameter-free jump.

Definition

For $\alpha \in \kappa^\kappa$, we say that $\Phi :\subseteq \kappa^{<\omega} \times \omega \to \kappa$ is *parameter-free κ-computable with finite queries to α* (*fq(α)-pf-κ-computable*) if there is a Σ_1 formula φ with some free variables, but without parameters from L_κ, such that for any $(\sigma, n) \in \mathbf{dom}(\Phi)$,

$$L_\kappa \models (\exists! \lambda)(\exists \tau \preceq \sigma)(\exists D \subseteq_{\mathrm{fin}} \kappa)\ \varphi(\tau, n, \lambda, D, \langle d, \alpha(d)\rangle_{d \in D}).$$

- Note: a parameter-free κ-computable function with finite queries always yields a partial continuous f function from κ^ω to κ^ω.
- We call such a function f *fq(α)-pf-κ-computable*.
- We write $y \leq^{\alpha}_{\kappa T} x$ if there is a fq(α)-pf-κ-computable function f such that $f(x) = y$.

Definition

The *short κ-jump* α^{sj} of $\alpha \in \kappa^\kappa$ is defined as follows:

$$\alpha^{sj}(e) = \begin{cases} \langle 1, \lambda_0, \langle D_0, \alpha(d)\rangle_{d\in D_0}\rangle \\ \qquad\qquad \text{if } L_\kappa \models (\exists!\lambda)(\exists D \subseteq_{\text{fin}} \kappa)\ \varphi_e(\lambda, D, \langle d, \alpha(d)\rangle_{d\in D}). \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

where φ_e is the e-th Σ_1-formula, and $\langle \lambda_0, D_0\rangle$ is the lexicographically first pair $\langle \lambda, D\rangle$ satisfying the required condition.

The *tall κ-jump* α^{tJ} of $\alpha \in \kappa^\kappa$ is the Σ_1-truth of L_κ with parameters, and finite queries to α.

- The short jump α^{sj} can be coded by an element of κ^ω.
- The tall jump α^{tJ} is an element of 2^κ.

- (The Friedberg Jump Inversion Theorem)
  $$(\forall x \geq_T 0')(\exists y)\ y' \equiv_T y \oplus 0' \equiv_T x.$$
- Relativized: $(\forall x \geq_T c')(\exists y \geq c)\ y' \equiv_T y \oplus c' \equiv_T x$.
- Equivalently: $(\forall x)(\exists y)\ (y \oplus c)' \equiv_T y \oplus c' \equiv_T x \oplus c'$.
- Equivalently:
  $$(\forall x)(\exists y) \quad x \leq_T (y \oplus c)' \leq_T^{c'} y \leq_T^{c'} x.$$

The Jump Inversion Theorem, relative to $c \in \kappa^\kappa$

$$(\forall x \in \kappa^\omega)(\exists y \in \kappa^\omega) \quad x \leq_{\kappa T} (y \oplus c)^{sj} \leq_{\kappa T}^{c^{tJ}} y \leq_{\kappa T}^{c^{tJ}} x.$$

Furthermore, all of the above κT-reductions are uniform.

In particular, $x \mapsto y$ is fq(c^{tJ})-pf-κ-computable.

- Define $sj^c(x) = (x \oplus c)^{sj}$ for $x \in \kappa^\omega$ and $c \in \kappa^\kappa$.
- Define $f^{\nleftarrow c} = f \circ (sj^c)^{-1}$.
- All computability-theoretic lemmas by K.-Montalbán hold with this.

- $^{\sqcup}S$: The set of all S-labeled discrete orders of cardinality $\leq \kappa$.
- **Tree**(S): The set of all S-labeled well-founded trees of cardinality $\leq \kappa$.
- $^{\sqcup}$**Tree**(S): The set of all S-labeled well-founded forests of cardinality $\leq \kappa$.

Theorem

Let Q be a BQO (better-quasi-order).

- The Wadge degrees of $\utilde{\Delta}^0_1(\kappa^\omega \to Q) \simeq {}^{\sqcup}Q$.
- The Wadge degrees of $\utilde{\Delta}^0_2(\kappa^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Wadge degrees of $\utilde{\Delta}^0_3(\kappa^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Wadge degrees of $\utilde{\Delta}^0_4(\kappa^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Wadge degrees of $\utilde{\Delta}^0_5(\kappa^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

Here we consider the quasi-Borel hierarchy in the sense of Martin (1990);
(A quasi-Borel set is a set constructed from open sets by taking complement, countable union, and open separated union.)

A reference for BQO-Wadge degrees on ω^ω:

[1] T. Kihara and A. Montalbán, *On the structure of the Wadge degrees of BQO-valued Borel functions*, **Trans. Amer. Math. Soc.** 371 (2019).

This work has also been motivated by Martin's conjecture, since (under $\mathbf{AD}^+$) there is one-to-one correspondence between

Wadge degrees (≈ reasonable pointclasses)

and

uniform degree invariant functions from $\equiv_T$ to $\equiv_m$ (≈ pointclass jumps)

See:

[2] T. Kihara and A. Montalbán, *The uniform Martin's conjecture for many-one degrees*, **Trans. Amer. Math. Soc.** 370 (2018).