Determined Borel sets and measurability

Linda Brown Westrick

Penn State University

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Borel sets and ATR_0

In Reverse Math, a Borel subset of 2^ω is a well-founded tree $T \subseteq \omega^{<\omega}$

- whose leaves are labeled with clopen sets
- whose inner nodes labeled $\cup$ or $\cap$

Let T_σ denote $\{\tau : \sigma\tau \in T\}$. Let $|T|$ denote the Borel set coded by T.

To determine if $X \in |T|$, inductively determine if $X \in |T_\sigma|$ for each $\sigma \in T$, starting with the leaves and working to the root (uses ATR_0)

Evaluation maps

How ATR_0 does it (in more detail)

Given $X \in 2^\omega$, an *evaluation map* for X in T is a function $f : T \to \{0, 1\}$

- if σ is a leaf, $f(\sigma) = 1$ iff $X \in |T_\sigma|$
- if σ is a $\cup$, then $f(\sigma) = 1$ iff for some n, $f(\sigma n) = 1$
- if σ is a $\cap$, then $f(\sigma) = 0$ iff for some n, $f(\sigma n) = 0$

We say $X \in |T|$ if and only if there is an evaluation map f for X in T such that $f(\lambda) = 1$.

Given T, a code for $|T|^c$ is obtained by swapping intersections and unions, and taking complements at the leaves.

Borel sets need ATR_0?

Proposition (Dhzafarov, Flood, Solomon & W). Over RCA_0, the statement "For every Borel set, either it or its complement is non-empty" implies ATR_0.

Recall, a set A has the *property of Baire* if there is an open set U such that $A \Delta U$ is meager.

Proposition (DFSW). Over RCA_0, ATR_0 is equivalent to the statement "Every Borel set has the property of Baire".

Proof. ($\Rightarrow$) The usual proof is by arithmetic transfinite recursion.
($\Leftarrow$) If a set has the Baire property, either it or its complement is nonempty. □

Nothing about this is satisfying

Transfinite mathematics below ATR_0

Previously in transfinite mathematics in systems weaker than ATR_0

- Hirst 2001. "With a few very interesting exceptions, most theorems of ordinal arithmetic are provable in RCA_0 or are equivalent to ATR_0"
- Greenberg & Montalbán 2008. "General statements about [well-founded trees, superatomic Boolean algebras, etc. are] equivalent to ATR_0, thereby implying the necessary use of ordinal ranks in the investigation of these classes"
  Tools include effective (Δ^0_1) transfinite recursion in ACA_0
- Simpson 2009. An ω-model M is an MTR-model (model of measure-theoretic regularity) if $M \models \mathrm{RCA}_0$ and every M-Borel set includes an M-F_σ set of the same measure.

The Borel Dual Ramsey Theorem

For $k \leq \omega$, let $(\omega)^k$ denote the partitions of ω into exactly k pieces.

Borel Dual Ramsey Theorem (Carlson & Simpson 1984). For any $k, \ell < \omega$, if

$$(\omega)^k = \bigcup_{i<\ell} C_i, \qquad C_i \text{ Borel}$$

partitions $(\omega)^k$ into ℓ-many Borel pieces, then there is an infinite partition $p \in (\omega)^\omega$ and a color $i < \ell$ such that for every $x \in (\omega)^k$ which coarsens p,

$$x \in C_i.$$

Does this imply ATR_0 for cheap reasons? No!
Notice,

$$\text{for all } x \in (\omega)^k, \quad x \in \bigcup_{i<\ell} C_i.$$

The evaluation maps for x in each C_i are posited in the hypothesis.

Determined Borel sets

Definition. A Borel set coded by T is *determined* if for every $X \in 2^\omega$, there is an evaluation map for X in T.

Note: In ACA_0, if an evaluation map exists, it is unique.

Definition. Let DPB be the statement "Every determined Borel set has the property of Baire".

Pathology removed: in RCA_0, for any determined Borel set, either it or its complement is nonempty.

Determined Borel sets have the property of Baire

Theorem 1 (Astor, Dzhafarov, Montalbán, Solomon & W). There is an ω-model of DPB in which ATR_0 fails.

Theorem 2 (ADMSW). Every ω-model of DPB is hyperarithmetically closed and contains a Δ^1_1-generic.

Borel Dual Ramsey Theorem

Fact. ATR_0 implies the Borel Dual Ramsey Theorem for 3-partitions.

Theorem 3 (ADMSW). There is an ω-model of the Borel Dual Ramsey Theorem for 3-partitions in which ATR_0 fails.

Theorem 4 (ADMSW). Every ω-model of the Borel Dual Ramsey Theorem for 3-partitions is hyperarithmetically closed.

Question. Is the Borel Dual Ramsey Theorem for 3-partitions a theory of hyperarithmetic analysis? (i.e. is $\Delta^1_1(X)$ a model of this principle for all X?)

Determined Borel sets are measurable

Definition. Let DBM be the statement "Every determined Borel set is measurable".

(There are a couple things *measurable* could mean, but they are equivalent over ACA_0)

Theorem 4. There is an ω-model of $\mathsf{DBM} + \mathsf{ACA}_0$ in which ATR_0 fails.

Theorem 5. Every ω-model of $\mathsf{DBM} + \mathsf{ACA}_0$ is hyperarithmetically closed and contains a Δ^1_1-random.

Strength of determined principles: upward

Proposition. Any ω-model of DPB or $\mathrm{DBM} + \mathrm{ACA}_0$ is hyperarithmetically closed.

Let us see why DPB implies ACA_0. Define clopen sets $C_{n,s}$ by

$$C_{n,s} = \begin{cases} [0^n 1] & \text{if } n \in \emptyset'_s \\ \emptyset & \text{otherwise.} \end{cases}$$

Let T be the code for $\bigcup_{n,s} C_{n,s}$. Then T is determined: for any X, it is uniformly X-computable whether $X \in C_{n,s}$, all that remains to make an evaluation map for X in T is to (non-uniformly) fill in the correct value at the root.

However, if U and V are open sets such that $|T| \Delta U$ and $|T|^c \Delta V$ are meager, then (U, V) computes $\emptyset'$.

How ATR_0 does the property of Baire

Consider the usual proof that every Borel set has the property of Baire.

Given T,

- If σ is a leaf, $|T_\sigma|$ is clopen. Let $U_\sigma = |T_\sigma|$, $V_\sigma = |T_\sigma|^c$
- If σ is a union, let $U_\sigma = \bigcup_n U_{\sigma n}$. (Then $|T_\sigma| \Delta U_\sigma$ is meager.) Use a jump to find V_σ.
- If σ is an intersection, let $V_\sigma = \bigcup_n V_{\sigma n}$. (Then $|T_\sigma|^c \Delta V_\sigma$ is meager.) Use a jump to find U_σ.

From this object $(U_\sigma, V_\sigma)_{\sigma \in T}$ one gets:

- $|T| \Delta U_\lambda$ is meager
- $|T| \Delta U_\lambda$ is contained in a meager set computable from (U_σ, V_σ)

Extended object: category

Do we need this extended object $(U_\sigma, V_\sigma)_{\sigma \in T}$?

Yes. In ACA_0, every determined Borel set has the property of Baire if and only if it has such an extended object.

To separate DPB from ATR_0, we need a model which has a different way of coming up with this object.

Polling Σ^1_1-generics

Theorem 1 (ADMSW). There is an ω-model of DPB in which ATR_0 fails.

Proof sketch. Let G be a Σ^1_1-generic. Write $G = \bigoplus_{n<\omega} G_i$. Let

$$\mathcal{M} = \bigcup_{n<\omega} \Delta^1_1\left(\bigoplus_{i<n} G_i\right).$$

Given T a code for a determined Borel set and $\sigma \in T$, let i be such that G_i is $\Sigma^1_1(T)$-generic. Define

$$U_\sigma = \bigcup_{p\in S_\sigma} [p] \qquad \text{where } S_\sigma = \{p \in 2^{<\omega} : \forall q \succeq p[\ qG_i \in |T_\sigma|\ ]\}$$

and similarly for V_σ. This $(U_\sigma, V_\sigma)_{\sigma\in T}$ works.

Extended object 1: measure via regularity

A set $B \subseteq 2^\omega$ is Lebesgue measurable if and only if there are $\mathbf{\Pi}^0_2$ sets A and C such that $A^c \subseteq B \subseteq C$ and $\mu(A \cup C) = 1$.

For the moment, we take this as our definition of measurability.

Given T, ATR_0 can produce a sequence $(A_\sigma, C_\sigma)_{\sigma \in T}$ satisfying for each σ,

$$A_\sigma^c \subseteq |T_\sigma| \subseteq C_\sigma, \qquad \mu(A_\sigma \cup C_\sigma) = 1$$

Again, to separate DBM and ATR_0 we need a model which has another way of producing such a sequence.

Polling Π^1_1-randoms

Theorem 4. There is an ω-model of DBM in which ATR_0 fails.

Proof sketch. Let G be a Π^1_1-random. Write $G = \bigoplus_{i<\omega} G_i$. Let

$$\mathcal{M} = \bigcup_{n<\omega} \Delta^1_1 \left(\bigoplus_{i<n} G_i \right).$$

Given T a code for a determined Borel set and $\sigma \in T$, let i be such that G_i is $\Pi^1_1(T)$-random. The columns of G_i supply us with many randoms to use for polling, but their responses yield a different kind of object.

Extended object 2: measure via integration

If $B \subseteq 2^\omega$ is Borel with code T, the function on $T \times 2^{<\omega}$ defined by

$$\sigma, p \mapsto \mu(|T_\sigma| \cap [p])$$

can be used to construct $\mathbf{\Pi}^0_2$ sets A and C such that $\mu(A \cup C) = 1$ and $A^c \subseteq B \subseteq C$.

Polling works.

Strength of determined principles: to the side

Theorem. Any ω-model of DPB (respectively $\mathsf{DBM} + \mathsf{ACA}_0$) contains a Δ^1_1-generic (respectively a Δ^1_1-random).

Proved using the method of decorating trees.

Corollary. Neither DPB nor DBM holds in *HYP*. Neither is a theory of hyperarithmetic analysis.

Thank you.

Preprints on arXiv:

- Dzhafarov, Flood, Solomon & Westrick. Effectiveness for the Dual Ramsey Theorem. arXiv: 1710.00070.
- Astor, Dzhafarov, Montalbán, Solomon & Westrick. The determined property of Baire in reverse math. arXiv: 1809.03940.