On the minimal size of a basis for uncountable linear order

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Basis for uncountable linear order

For a class of structures $\mathcal{K}$, a subclass $\mathcal{B}$ is a basis for $\mathcal{K}$ if every structure in $\mathcal{K}$ has a substructure that is in $\mathcal{B}$. We will study the basis for $\mathcal{K}$ when $\mathcal{K}$ is the stucture of

- uncountable linear order

or related stuctures inculding

- Aronszajn lines
- Aronszajn tree
- ω_1-dense subset of reals

Some examples of uncountable linear orders:

- uncountable subsets of real line
- ω_1
- ω_1^*
- Suslin line

Notation: Those uncountable linear orders which do not contain uncountable separable suborders or copies of ω_1 or ω_1^* are called Aronszajn lines.

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Suslin line is a Aronszajn line. The existence of Suslin line is independent of ZFC. However, Shelah manages to construct a special type of Aronszajn line under ZFC.

Theorem (Shelah)

There is a Countryman line.

Here an uncountable linear order $(X, <)$ is called a Countryman line if $(X^2, <^2)$ can be decomposed into ω many chain. Here

$$(x_0, y_0) <^2 (x_1, y_1) \iff x_0 < x_1 \wedge y_0 < y_1.$$

Minimal size of basis for linear order

Theorem (Baumgarterner)

Assume PFA. Any two ω_1-dense subsets of reals are isomorphic.

Theorem (Moore)

Under PFA, for any Countryman line C, all Aronszajn line contains C or C^* .

Moore's theorem can be seperated into following substatement:

- any two normal Aronszajn trees are club isomorphic under PFA.(Shelah)
- MA_{ω_1} implies for any special countryman tree T, there is a 2 element basis for suborders of Countryman lines with partition tree T. (Todorcevic)
- every Aronszajn line contains a Countryman subline under PFA.

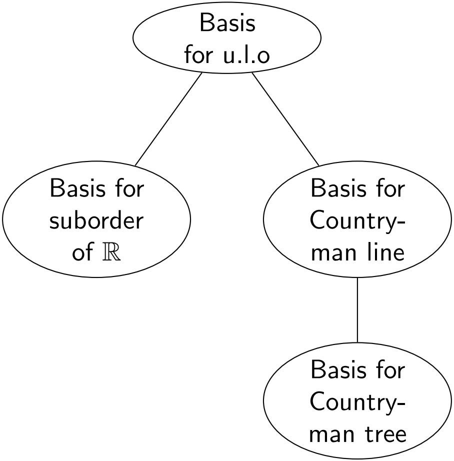
In conclusion, under PFA, there is a 5-element basis of uncountable linear order.

To the contrast, the minimal size of basis could also be huge.

Theorem (Sierpinski)

Under CH, there is no basis for the uncountable separable linear orders of cardinality less than 2^{ω_1}.

The following diagram describes the constrution of basis:



Coherent trees

Definition (Todorcevic)

An Aronszajn tree $T \subset \omega^{<\omega_1}$ is *coherent* if for any $s, t \in T$,
$D_{s,t} = \{\alpha < ht(s), ht(t) : s(\alpha) \neq t(\alpha)\}$ is finite.

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Definition (Todorcevic)

For a coherent tree T, $\mathcal{I}_T$ is the ideal consists of $\Gamma \subset \omega_1$ disjoint from $\Delta(X)$ for some $X \in [T]^{\omega_1}$ where $\Delta(X) = \{\min(D_{s,t}) : s \perp t \text{ are in } X\}$.

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$\mathcal{I}_T$ is an ideal for any coherent tree T.

Simple facts about $\mathcal{I}_T$

Fact

If $\Gamma \notin \mathcal{I}_T$ and $\mathcal{P}$ is a forcing with property (K), then $\Vdash_{\mathcal{P}} \Gamma \notin \mathcal{I}_T$.

FA$(\mathcal{K})$ asserts that if $\mathcal{P}$ is a forcing with property (K) and $\mathcal{D}$ is a clooection of ω_1 many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$.
FA$(\mathcal{K})$ implies that

- any Countryman line can be embedded into any of its uncountable subline,
- for a special coherent tree T, the basis for Countryman lines that have partition tree T has size $|\mathcal{P}(\omega_1)/\mathcal{I}_T|$.

Proposition (Todorcevic)

MA_{ω_1} implies that $\mathcal{I}_T$ is a maximal ideal, i.e., $|\mathcal{P}(\omega_1)/\mathcal{I}_T| = 2$.

A possible solution

Lemma (Key Lemma)

The minimal size of a basis for uncountable linear order is $2^n + 3$ *if the following conditions are satisfied:*

1. *there is a one element basis for uncountable set of reals,*
2. $FA(\mathcal{K})$,
3. *any two normal Aronszajn trees are club isomorphic,*
4. $|\mathcal{P}(\omega_1)/\mathcal{I}_T| = 2^n$,
5. *every Aronszajn line contains a Countryman subline.*

PFA($\overset{\rightharpoondown}{\Gamma}$)

Suppose $\overset{\rightharpoondown}{\Gamma}$ is a partition of ω_1 and T is a special coherent tree.

PFA($\overset{\rightharpoondown}{\Gamma}$) asserts that $\overset{\rightharpoondown}{\Gamma} \cap \mathcal{I}_T = \emptyset$, and if $\mathcal{P}$ is a proper forcing that does not force $\overset{\rightharpoondown}{\Gamma} \cap \mathcal{I}_T \neq \emptyset$ and $\mathcal{D}$ is a collection of ω_1 many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$.

Lemma

PFA($\overset{\rightharpoondown}{\Gamma}$) implies all conditions in the Key Lemma are satisfied.

It also can be verified that PFA($\overset{\rightharpoondown}{\Gamma}$) implies MRP.

The consistency

Theorem

Suppose that $\Gamma \notin \mathcal{I}_T$ and $\mathcal{P}$ is a countable support iterated forcing such that each iterant is proper and force $\Gamma \notin \mathcal{I}_T$, then $\mathcal{P}$ doesnot force Γ to be in $\mathcal{I}_T$.

Corollary

If there is a supercompact cardinal and $\vec{\Gamma}$ is a partition of ω_1, then there is a model satisfying PFA($\vec{\Gamma}$).

Corollary

If there is a supercompact cardinal, then for each $n < \omega$, there is a model in which the basis of uncountable linear orders has size $2^n + 3$.

More possible sizes

$\overrightarrow{T}$ is a collection of complete special coherent trees.
PFA($\overrightarrow{T}$) asserts that $\overrightarrow{T}$ are pairwise non-club-isomorphic and if $\mathcal{P}$ is a proper forcing that does not force any two trees in $\overrightarrow{T}$ to be club-isomorphic and $\mathcal{D}$ is a collection of ω_1 many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$.

More possible sizes

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Theorem

For any $n < \omega$, Con(a supercompact cardinal) implies Con(PFA($\overrightharpoon{T}$) for some finite $\overrightharpoon{T}$) such that $|\overrightharpoon{T}| = n$.

Basis for coherent trees

Fix some $\vec{T}$ of size $n < \omega$.

Theorem

Assume PFA($\vec{T}$). Then

- MA_{ω_1}.
- *Every coherent tree contains a subtree club-isomorphic to one of $\vec{T}$.*
- *The basis for Countryman lines has size $2n$.*

It is unknown whether PFA($\vec{T}$) implies that every Aronszajn tree has a subtree club-isomorphic to a coherent tree.

Basis for subsets of reals

Theorem (Abraham, Rubin, Shelah)

For any $n < \omega$, Con(MA+the minimal size of a basis for uncountable subset of reals is n).

The forcing in the proof is a finite support iterated c.c.c forcing. No good forcing theory for the class of n element basis preserving proper forcing is known.

2 element basis for reals

When $n = 2$, the following stronger property is easy to use.

Definition

Let $A \subseteq R$ be uncountable, A is called an increasing set, if in every uncountable set of pairwise disjoint finite sequences from A, there are two sequences $(a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})$ having the same length such that $a_i < b_i$ for any $i \leq n - 1$

Proposition (Abraham,Shelah)

Assume MA_{ω_1} and there is an increasing set A, then the minimal size of a basis for uncountable subset of reals is 2.

Forcing preserves increasing set

PFA(inc(A)) asserts that for A is increasing and if $\mathcal{P}$ is a proper forcing that forces A is increasing and $\mathcal{D}$ is a collection of ω_1 many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$.

Theorem

Assume PFA(inc(A)). Then

- MA_{ω_1}.
- *any two normal Aronszajn trees are club isomorphic.*
- *every Aronszajn line contains a Countryman subline.*
- *the minimal size of a basis for uncountable subset of reals is* 2.

However, no good iteration theory for $inc(A)$ is known. In particular, it is unknown how to force $PFA(inc(A))$ using iterated forcing. We use the $\mathbb{P}_{max}$ machinery instead.

$\mathbb{P}_{max}$ variant for increasing set

Following Woodin's generalized version of $\mathbb{P}_{max}$ forcing, we define the variant $\mathbb{P}^{inc}_{max}$ as follows: The partial order $\mathbb{P}^{inc}_{max}$ consists of all pairs $\langle (M, I), a, K\rangle$ such that

1. M is a countable transitive model of ZFC^-,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. M think a is increasing subset of reals.
5. $K \in M$ and K is a set of pairs $(\langle (N, J), b, E\rangle, j)$ such that
   - $\langle (N, J), b, y, E\rangle \in \mathbb{P}^{\Gamma}_{max} \cap H(\omega_1)^M$,
   - j is an iteration of (N, J) of length ω_1^M such that $j(J) = I \cap j(N)$ and $j(b) = a$,
   - $j(E) \subset K$,

   with the property that for each $p \in \mathbb{P}^{\Gamma}_{max}$ there is at most one j such that $(p, j) \in X$.

Say $\langle (M', I'), a', K'\rangle < \langle (M, I), a, K\rangle$ if there exists a j such that $(\langle (M, I), a, K\rangle, j) \in K'$.

By verifying the iteration lemma, we have:

Proposition

$\mathbb{P}^{\Gamma}_{max}$ *forces the following:*

- A_G *is increasing.*
- NS_{ω_1} *is saturated, or* $Sat(NS_{\omega_1})$ *holds.*
- MA_{ω_1}.
- *There is a 2 element basis for countryman line.*
- *If ground model satisfies* $V = L(P(\mathbb{R})) + AD_{\mathbb{R}}$*, then* $BPFA(inc(A))$ *holds.*

Konig–Larson-Moore-Velickovic shows that $Sat(NS_{\omega_1}) + BPFA$ implies every Aronzajn line contains a Countryman line. By analysing the forcing poset used in their proof, $Sat(NS_{\omega_1}) + BPFA(inc(A))$ also works for the purpose. Putting these together, we have

Theorem

Assuming $V = L(P(\mathbb{R})) + AD_{\mathbb{R}}$, $\mathbb{P}^{\Gamma}_{max}$ *forces the minimal size of basis for uncountable linear order is* 6.

Question

As a natural question, we are curious about basis of arbitrary size.

Question

For any fixed $n \geq 5$, is it consistent that the minimal size of basis for uncountable linear order is n?

Thank you for your attention!