Are all singular cardinals born equal ? The case of $\aleph_\omega$ and $\aleph_{\omega^2}$

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3. "Weak Chang's conjecture" : If $\mathcal{A} = \langle \kappa, R, \ldots \rangle$ is a structure in a countable language such that R is a unary predicate such that $|R| < \kappa$ then there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ such that $|\mathcal{B}| < \kappa$ and $|R \cap \mathcal{B}| < \kappa$.

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A topological example

Compactness for a topological space being collection-wise Hausdorff: A topological space X is collection-wise Hausdorff if every discrete subset can be separated . Namely if $Y \subseteq X$ is discrete , then there is a mutually disjoint family of open sets $\{y | y \in Y\}$ such that for $y \in Y$ $y \in U_y$. Suppose that X is a topological space of cardinality κ such that every subspace of cardinality $< \kappa$ is collection-wise Hausdorff . Is X collection-wise Hausdorff? In order to avoid trivial counter example we need to assume that X is "locally small". e.g. "Every point has a neighbourhood of cardinality whose successor is less than κ.

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In this talk we are mainly interested in the case of successors of singulars . In particular $\aleph_{\omega+1}$ and it neighbours.

Implications between compactness properties

Theorem

Let κ be a regular cardinal. Suppose that there a non reflecting stationary set $S \subseteq \kappa$ such that for $\alpha \in S$ $\mathrm{cf}(\alpha) = \omega$. Then:

1. *There is a family of countable sets $\mathcal{F}$ of cardinality κ such that every smaller cardinality subfamily has a transversal, but $\mathcal{F}$ fails to have a transversal.*

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2. *There is a (an Abelian) group G of cardinality κ such that every subgroup of smaller cardinality is free but G is not free.*
3. *There is a topological space of cardinality κ , which is locally countable ("every point has a countable neighbourhood") such that every subspace of smaller cardinality is collection-wise Hausdorff but the space is not collection-wise Hausdorff.*

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Hence if $\kappa = \lambda^+$ where λ is singular implies that $\square_\lambda$ fails. Hence the consistency strength of a successor of singular having any of the above compactness properties is rather large.

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The good news

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Theorem (Shelah)

Assuming he consistency of supercompact cardinal it is consistent that every topological space X of cardinality $\aleph_{\omega+1}$ which is locally countable, if every subspace of smaller cardinality is collection wise Hausdorff, then X is collection wise Hausdorff.

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Theorem (M.-Shelah)

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The same is true for all $\aleph_{\omega \cdot n+1}$ for all $n < \omega$.

Combining stationary reflection and Chang's conjecture

Definition (M.-Shelah)

Let $\kappa = \lambda^+$ where λ is singular. We say that "Delta" reflection holds at δ , denoted by Δ_λ, if for every structure (in a countable language) $\mathcal{A} = \langle \kappa, <, S, \lambda, R_0 \ldots \rangle$ such that S is a stationary subset of κ and for every $\delta < \lambda$ there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ such that $|\mathcal{B}| < \kappa$,$\delta \subseteq \mathcal{B}$ $|\mathcal{B} \cap \lambda| < |\mathcal{B}|$ and $S \cap \sup(\mathcal{B})$ is stationary in $\sup(\mathcal{B})$.

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Note that Δ_λ is preserved by forcings of size less than λ.

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Unfortunately Δ_λ does not imply the tree property at λ^+. (M.- L. Fontanella)

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Theorem (M+Shelah)

Assuming the consistency of ω supercompacts it is consistent that $\Delta_{\aleph_{\omega^2}}$ holds. So in the resulting model $\aleph_{\omega^2+1}$ is compact for all the properties we considered . except possibly the tree property.

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Assuming the consistency of ω *many supercompacts it is consistent that* $\Delta_{\aleph_{\omega^2}}$ *together with the tree property at* $\aleph_{\omega^2+1}$*. So in the resulting model* $\aleph_{\omega^2+1}$ *has all the compactness properties we considered (and many more) .*

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A slight modification of the above construction gets a model in which $\Delta_{\aleph_{\omega^2}}$ holds but $\aleph_{\omega^2+1}$ fails the tree property.(So $\Delta_{\aleph_{\omega^2}}$ and the tree property are independent of each other.)

Why is $\aleph_{\omega^2+1}$ different from $\aleph_{\omega+1}$?

Definition

Let λ be a singular cardinal and $\mu = \text{cf}(\lambda)$. A λ^+ scale is a sequence $\langle \kappa_i | i < \mu \rangle$ of regular cardinals less than λ , cofinal in λ and a sequence $\langle g_\alpha | \alpha < \lambda^+ \rangle$ which is increasing and cofinal in $\langle \prod_{i<\mu} \kappa_i, \precsim \rangle$ where $f \precsim g$ iff $\{ i < \mu | f(i) > g(i) \}$ is bounded in μ.

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Theorem (Shelah)

For every singular cardinal λ there exists a λ^+ scale.

The good and the bad

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Let λ be a singular cardinal and $\vec{g} = \langle g_\alpha | \alpha < \lambda^+ \rangle$ a λ^+ scale.
$D \subseteq \vec{g}$ is disjointifable if there is function $d : D \to \mu$ such that for all $\alpha < \beta$ in D and $i > \max(d(\alpha), d(\beta))$ $g_\alpha(i) < g_\beta(i)$.

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3. *There is a (an Abelian) group G of cardinality λ^+ such that every smaller cardinality subgroup is free but G is not free.*
4. *There is a topological space X which is locally of cardinality $\leq \aleph_1$ such that every smaller cardinality subspace is collection wise Hausdorff but X fails to be collection wise Hausdorff.*

Finer analysis of badness

Definition

Let $\vec{g}$ be a λ^+ scale and let $\eta < \lambda$ be regular. $\vec{g}$ is η- good if the set of $\alpha < \lambda^+, \mathrm{cf}(\alpha) = \eta$ such that α is NOT good for $\vec{g}$ is non stationary in λ^+.

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Lemma

Let $\vec{g}$ be a λ^+ which η- good for some regular $\eta < \lambda$. Then there is a structure $\mathcal{A} = \langle \lambda^+, \lambda, R_0, \ldots \rangle$ such that for every $\mathcal{B} \prec \mathcal{A}$ where $|\mathcal{B}| < \lambda$ then either $|\mathcal{B} \cap \lambda| = |\mathcal{B}|$ or $\mathrm{cf}(\sup(\mathcal{B})) \neq \eta$.

Theorem (M.+Shelah)

Let $\vec{g}$ *be* λ^+ *scale which is* ρ *good for every regular* $\rho, \eta \leq \rho < \lambda$ *for some then*

1. *There is a family* $\mathcal{F}, |\mathcal{F}| = \lambda^+$ *of sets of whose cardinality is* $\leq \max(\mathrm{cf}(\lambda)^+, \eta)$ *such that every smaller cardinality has a transversal but* $\mathcal{F}$ *fails to have a transversal.*

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In the above theorem, if $\leq \max(\mathrm{cf}(\lambda)^+, \eta) < \aleph_\omega$ then we can assume that $\mathcal{F}$ is made up of countable sets , that the space X is locally of cardinality $\leq \omega_1$ and that there is a (an Abelian) group G of cardinality λ^+ every smaller cardinality subgroup is free , while G is not free.

Theorem

Let $\vec{g}$ *be* $\aleph_{\omega+1}$ *scale . Then every* $\alpha < \aleph_{\omega+1}$ *such that* $\mathrm{cf}(\alpha) \geq \aleph_4$ *is good for* $\vec{g}$*.*

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Suppose that every stationary subset of $\aleph_{\omega+1}$ reflects, then every $\aleph_{\omega+1}$ scale is good.

Corollary

Every Chang's conjecture for $\aleph_{\omega+1}$ is incompatible with stationary reflection for $\aleph_{\omega+1}$.

Is the tree property separates $\aleph_{\omega+1}$ from $\aleph_{\omega^2+1}$

In all the models in which $\aleph_{\omega+1}$ has the tree property, no scale is good. So in these models there is a non reflecting stationary subset of $\aleph_{\omega+1}$.

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Theorem (Sinapova)

Assuming the consistency of ω many supercompacts. It is consistent that $\aleph_{\omega^2+1}$ has the tree property, $\aleph_{\omega^2}$ is strong limit and $2^{\aleph_{\omega^2}} > \aleph_{\omega^2+1}$. Hence $\aleph_{\omega^2+1}$ has the tree property and it caries a good scale.

Conjecture

If $\aleph_{\omega+1}$ has the tree property then no $\aleph_{\omega+1}$ scale is good. In particular , assuming that $\aleph_{\omega+}$ has the tree property , then it satisfies the singular cardinals hypothesis and it fails stationary reflection.

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Compare with the fact that the tree property at $\aleph_{\omega^2}$ is consistent with $\Delta_{\aleph_{\omega^2}}$. It is it is consistent with stationary reflection.

The survival of the tree property under small forcing

Theorem (Hayut-M.)

Assuming the consistency of ω many supercompact cardinals then

1. *it is consistent to have a model in which $\aleph_{\omega+1}$ has the tree property , but if we force with the Levy collapse of ω_1 then there is a special Aronszajn tree on $\aleph_{\omega+}$.*

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Assuming the consistency of one supercompact then it is consistent that $\aleph_{\omega+1}$ *have simultaneously the tree property and the Chang's conjecture* $(\aleph_{\omega+1}, \aleph_\omega) \Rightarrow (\aleph_1, \aleph_0)$*. Of course it can not have simultaneously stationary reflection.*

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$\aleph_{\omega^2+1}$ is very different from $\aleph_{\omega+1}$

Dear Hugh and Ted
Thank you for the wonderful mathematics but also thanks for making the Singapore logic summer program being the great sucess it is !