The Implicitly Constructible Universe

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Overview

Imp is a model of set theory defined by Hamkins and Leahy.

L is constructed by iterating definability through the ordinals.

Imp is constructed by iterating *implicit* definability through the ordinals.

Imp is intermediate between **L** and **HOD**:

$$\mathbf{L} \subseteq \mathbf{Imp} \subseteq \mathbf{HOD}$$

Joint work with Joel Hamkins answers some questions about the possible nature of **Imp**.

I will also discuss a further result.

Defining $\mathbf{L}$

A set $X \subseteq M$ is *definable* over M if there are a formula $\varphi(x, y_1, \ldots, y_n)$ in the language of ZF, and elements $a_1 \ldots, a_n \in M$, such that

$$X = \{x \in M \mid (M, \in) \models \varphi(x, a_1, \ldots, a_n)\}.$$

$\mathbf{L}_0 = \emptyset$;

$\mathbf{L}_{\alpha+1} = \{X \mid X \text{ is definable over } \mathbf{L}_\alpha\}$;

$\mathbf{L}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha$ for limit λ;

$\mathbf{L} = \bigcup_{\alpha \in OR} \mathbf{L}_\alpha$.

Defining **Imp**

A set $X \subseteq M$ is *implicitly definable* over M if there are a formula $\varphi(y_1, \ldots, y_n)$ in the language of ZF with an additional predicate, and elements $a_1, \ldots, a_n \in M$, such that X is the unique subset of M for which

$$(M, X, \in) \models \varphi(a_1, \ldots, a_n).$$

$\mathbf{Imp}_0 = \emptyset$;

$\mathbf{Imp}_{\alpha+1} = \{X \mid X \text{ is implicitly definable over } \mathbf{Imp}_\alpha\}$;

$\mathbf{Imp}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{Imp}_\alpha$ for limit λ;

$\mathbf{Imp} = \bigcup_{\alpha \in OR} \mathbf{Imp}_\alpha$.

Defining **HOD**

A set X is *ordinal definable* if if there are a formula $\varphi(x, y_1, \ldots, y_n)$ in the language of ZF, and ordinals $\alpha_1 \ldots, \alpha_n$, such that

$$X = \{x \mid \varphi(x, \alpha_1, \ldots, \alpha_n)\}.$$

X is *hereditarily ordinal definable* if every member of the transitive closure of X is ordinal definable.

$\mathbf{HOD} = \{X \mid X \text{ is hereditarily ordinal definable}\}$.

A little recursion theory

$\mathbf{L}_\omega = \mathbf{Imp}_\omega = HF$.

$\mathbf{L}_{\omega+1}$ contains exactly the arithmetic sets, those definable by a formula in the language of arithmetic.

$\mathbf{Imp}_{\omega+1}$ contains all the Π^0_1 singletons, since "X is a branch through the e^{th} recursive tree" can be expressed arithmetically.

By Shoenfield absoluteness, for any $\alpha \leq \omega_1^{\mathbf{L}}$, we have

$$\mathbf{Imp}_\alpha = (\mathbf{Imp}_\alpha)^{\mathbf{L}}$$

and so

$$\mathbf{Imp}_{\omega_1^{\mathbf{L}}} = \mathbf{L}_{\omega_1^{\mathbf{L}}}.$$

Results fromHamkins and Leahy

Imp is an inner model of ZF.

$\mathbf{L} \subseteq \mathbf{Imp} \subseteq \mathbf{HOD}$

The following are consistent with $ZFC + V \neq \mathbf{L}$:

(1.) $\mathbf{Imp} = \mathbf{L}$.

(2.) $\mathbf{Imp} = V$.

(3.) $\mathbf{L} \subset \mathbf{Imp} \subset V$.

Questions (Hamkins and Leahy)

Which of the following are consistent with *ZFC*?

(1.) $\mathbf{Imp} \neq \mathbf{HOD}$.

(2.) $\mathbf{Imp} \models \neg CH$.

(3.) $\mathbf{Imp}^{\mathbf{Imp}} \neq \mathbf{Imp}$.

(4.) $\mathbf{Imp} \models \neg AC$.

Joint work with Hamkins shows (1), (2) and (3) are consistent.
I will outline the proof for (3) and an approach to (4).

Tools (Hamkins and Leahy)

Suppose G is $\mathbb{P}$-generic over M.

- If $\mathbb{P}$ is almost homogeneous, then in $M[G]$

$$\mathbf{Imp} \subseteq M.$$

($\mathbb{P}$ is almost homogeneous if for every $p, q \in \mathbb{P}$ there are extensions $\overline{p}, \overline{q}$ and an automorphism of $\mathbb{P}$ taking $\overline{p}$ to $\overline{q}$.)

- If G is the unique $\mathbb{P}$-generic, then in $M[G]$

$$M \subseteq \mathbf{Imp} \implies \mathbf{Imp} = M[G].$$

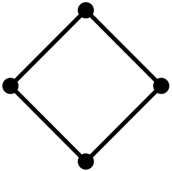
We will use this, playing off homogeneity against rigidity,
to obtain a model of $\mathbf{Imp}^{\mathbf{Imp}} \neq \mathbf{Imp}$.

Our tools

We force over $M \models V = \mathbf{L}$ with a combination of products and iterations of Sacks forcing.

Sacks forcing $\mathbb{S}$ adds a minimal degree of constructibility ($\mathbf{L}$-degree).

Forcing with the product $\mathbb{S} \times \mathbb{S}$ gives $\mathbf{L}$-degrees



Forcing with the iteration $\mathbb{S} * \mathbb{S}$ gives $\mathbf{L}$ degrees

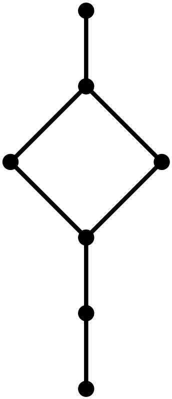
We can combine products and iterations to get complicated structures.

Forcing with

$$\mathbb{S} * \mathbb{S} * (\mathbb{S} \times \mathbb{S}) * \mathbb{S}$$

gives this ordering of **L**-degrees.

This is a finite tower of lines and diamonds.



Coding

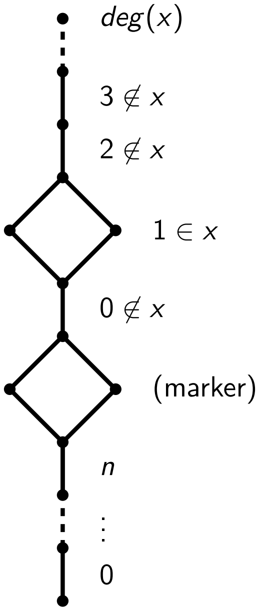
We will use this to produce an ω-length iteration
that adds a tower of lines and diamonds of height $\omega + 1$,

with top point $deg(x)$
(where x is the join of the individual generic reals),

such that a number $n \in \omega$ and the real x itself
are coded into the degree structure below x.

Such an x is a *self-coding real* with base n.

L-degrees below a self-coding real x with base n:



Being in **Imp** is not absolute

Since the generic is coded into the **L**-degrees,
the forcing $\mathbb{Q}_n$ to add a self-coding real with base n
adds to $M \models V = \mathbf{L}$ a unique generic real g.

Therefore, in $M[g]$ we have $\mathbf{Imp} = M[g]$.

If $\mathbb{P}$ is a countable support product of ω_1-many copies of $\mathbb{Q}_n$,
$\mathbb{P}$ is almost homogeneous.

Therefore, in $M[G]$ we have $\mathbf{Imp} = M$.

$M \models V = \mathbf{L}$

If $\mathbb{P}$ is a countable support product of ω_1-many copies of $\mathbb{Q}_n$, G is generic for $\mathbb{P}$, and g is generic for one copy of $\mathbb{Q}_n$, then $M \subseteq M[g] \subseteq M[G]$.

In $M[g]$, we have $g \in \mathbf{Imp}$, because of the uniqueness of g.

In $M[G]$, we have $g \notin \mathbf{Imp}$, because g is no longer unique.

This is a simple example of the phenomenon we will exploit.

The forcing

Add to $M \models V = \mathbf{L}$ via countable support product forcing:

$$
\begin{array}{ccccccccc}
 & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & & b_{0,\alpha} & b_{1,\alpha} & b_{2,\alpha} & \cdots & b_{n,\alpha} & b_{n+1,\alpha} & \cdots \\
 & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & & b_{0,3} & b_{1,3} & b_{2,3} & \cdots & b_{n,3} & b_{n+1,3} & \cdots \\
 & & b_{0,2} & b_{1,2} & b_{2,2} & \cdots & b_{n,2} & b_{n+1,2} & \cdots \\
 & & b_{0,1} & b_{1,1} & b_{2,1} & \cdots & b_{n,1} & b_{n+1,1} & \cdots \\
a & & b_{0,0} & b_{1,0} & b_{2,0} & \cdots & b_{n,0} & b_{n+1,0} & \cdots
\end{array}
$$

Sacks generic

self-coding with base n

ω_1-many for each n

The submodel

Define $H = (a,\ b_{0,0},\ b_{1,0},\ \ldots,\ b_{n,0}\ldots)$ and $h = \bigoplus H$.

$$
\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & b_{0,\alpha} & b_{1,\alpha} & b_{2,\alpha} & \cdots & b_{n,\alpha} & b_{n+1,\alpha} & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
 & b_{0,3} & b_{1,3} & b_{2,3} & \cdots & b_{n,3} & b_{n+1,3} & \cdots \\
 & b_{0,2} & b_{1,2} & b_{2,2} & \cdots & b_{n,2} & b_{n+1,2} & \cdots \\
 & b_{0,1} & b_{1,1} & b_{2,1} & \cdots & b_{n,1} & b_{n+1,1} & \cdots \\
\boxed{a} & \boxed{b_{0,0}} & \boxed{b_{1,0}} & \boxed{b_{2,0}} & \cdots & \boxed{b_{n,0}} & \boxed{b_{n+1,0}} & \cdots
\end{array}
$$

Let $K = H^\frown(b_{n,\alpha} \mid 0 < \alpha < \omega_1 \ \&\ n \notin h)$. Define $N = M[K]$.

$$
\begin{array}{cccccccc}
 & \vdots & & \vdots & & & \vdots & \\
 & b_{0,\alpha} & & b_{2,\alpha} & \cdots & & b_{n+1,\alpha} & \cdots \\
 & \vdots & & \vdots & & & \vdots & \\
 & b_{0,3} & & b_{2,3} & \cdots & & b_{n+1,3} & \cdots \\
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a & b_{0,0} & b_{1,0} & b_{2,0} & \cdots & b_{n,0} & b_{n+1,0} & \cdots \\
 & & & & & & & \\
 & & 1 \in h & & & n \in h & & \\
 & 0 \notin h & & 2 \notin h & & & n+1 \notin h &
\end{array}
$$

Defining **Imp** in N

By homogeneity properties, the other $b_{n,\alpha}$ are not in N.

In N, $b_{n,0}$ is the unique self-coding real with base n ifff $n \in h$.

$$
\begin{array}{cccccccc}
 & \vdots & & \vdots & & & \vdots & \\
 & b_{0,\alpha} & & b_{2,\alpha} & \cdots & & b_{n+1,\alpha} & \cdots \\
 & \vdots & & \vdots & & & \vdots & \\
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\end{array}
$$

Define **Imp** in N:

- ○ $\mathbf{Imp}_{\omega_1} = \mathbf{L}_{\omega_1}$.
- $b_{n,0} \in \mathbf{Imp}_{\omega_1+1} \iff n \in h$.
- $h \in \mathbf{Imp}_{\omega_1+2}$.

$\mathbf{Imp} = M[H]$. (The almost-homegeneity of product forcing guarantees nothing extraneous is in **Imp**.)

Finishing the proof

In N we have $\mathbf{Imp} = M[H]$. To show $N \models \mathbf{Imp}^{\mathbf{Imp}} \neq \mathbf{Imp}$, show

$$\mathbf{Imp}^{M[H]} \neq M[H].$$

$M[H] = M[a, (b_{n,0} \mid n \in \omega)] = M[(b_{n,0} \mid n \in \omega)][a]$,

where a is generic over $M[(b_{n,0} \mid n \in \omega)]$ for $\mathbb{S}^M$.

But $\mathbb{S}^M$ is almost homogeneous. Thus, in $M[H]$, we have

$$\mathbf{Imp} \subseteq M[(b_{n,0} \mid n \in \omega)] \qquad a \notin \mathbf{Imp}.$$

Hence $\mathbf{Imp}^{M[H]} \neq M[H]$. That is, $\left(\mathbf{Imp}^{\mathbf{Imp}}\right)^N \neq (\mathbf{Imp})^N$.

Imp $\not\models$ AC: The forcing

Iterate $\mathbb{S}$ and $\mathbb{S} \times \mathbb{S}$ transfinitely.

For $\alpha < \omega_2$, the forcing $\mathbb{P}_\alpha$ adds a self-coding α-tower (height $\omega \cdot \alpha + \omega$) topped by a real.
The generic, hence the real, is unique.

Force over $M \models V = \mathbf{L}$ with a countable support product:

For each $\alpha < \omega_2$, have ω_2-many copies of $\mathbb{P}_\alpha$, adding α-towers topped by reals $g_{\alpha,\beta}$.

Also add ω_2-many Sacks reals a_β for $\beta < \omega_2$.

The submodel

$H = (a_0, \dots, a_\beta, \dots g_{0,0}, \dots, g_{\alpha,0} \dots)$
includes the Sacks reals and an α-tower top for each α.

N is $M[H]$ plus the remaining α-tower tops for $\alpha \notin X$.

The set X codes everything in $\mathbf{L}_{\omega_2}[H]$.

For $\alpha < \omega_2$ we have $|\mathbf{Imp}_\alpha| < \omega_2$, so elements of $\mathbf{L}_{\omega_2}[H]$ appear in $\mathbf{Imp}_{\omega_2}$, but nothing larger does.

$\mathbf{Imp}_{\omega_2} = \mathbf{L}_{\omega_2}[H]$.

Proof outline

To see no choice function for the set of minimal degrees appears in **Imp** at any later stage:

For any conditions p, q that differ only on the Sacks forcing part, there is an automorphism φ taking p to q such that

$$\mathbf{Imp}_{\omega_2} = \mathbf{L}_{\omega_2}[H] = \mathbf{L}_{\omega_2}[\varphi H].$$

For all $\gamma \geq \omega_2$, there is a term for $\mathbf{Imp}_\gamma$,
with the same interpretation relative to any such φH.

Suppose that Y is a set of ordinals coding a choice function on minimal degrees and $Y \in \mathit{Imp}$,
so $Y \in \mathbf{Imp}_{\gamma+1}$ for some $\gamma \geq \omega_2$.

Then Y is the unique solution over $\mathbf{Imp}_\gamma$ to some formula with parameters from $\mathbf{Imp}_\gamma$.

There are an ordinal $\rho < \omega_2$ and a term for Y that has the same realization relative to any φH, provided φ fixes a_β for all $\beta < \rho$.

This implies $Y \in M[(g_{\alpha,0}, \ldots, g_{\alpha,\beta}, \ldots)][(a_\beta \mid \beta < \rho)]$.

But a_β for $\beta \geq \rho$ is not in $M[(g_{\alpha,0}, \ldots, g_{\alpha,\beta}, \ldots)][(a_\beta \mid \beta < \rho)]$, a contradiction.

References

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