Measurable Hall's theorem for actions of abelian groups

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Definition

Suppose Γ is a group acting on a space X. Two subsets $A, B \subseteq X$ are Γ-*equidecomposable* if there are partitions

$$A_1, \ldots, A_n, \quad B_1, \ldots, B_n$$

of both sets

$$A = \bigcup_i A_i \quad B = \bigcup_i B_i$$

such that

$$\gamma_i A_i = B_i$$

for some $\gamma_1, \ldots, \gamma_n \in \Gamma$.

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Banach–Tarski paradox

The Banach–Tarski paradox says that the unit ball and two copies of the unit ball in $\mathbb{R}^3$ are $\mathrm{Iso}(\mathbb{R}^3)$-equidecomposable.

Fact (Banach)

For Γ amenable group, preserving a probability measure μ on X and two measurable sets A, B if A and B are equidecomposable, then $\mu(A) = \mu(B)$

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Question (Tarski, 1925)

Are the unit square and the unit disc equidecomposable using isometries on $\mathbb{R}^2$?

Theorem (Laczkovich, 1990)

If $A, B \subseteq \mathbb{R}^n$ are bounded, measurable such that $\mu(A) = \mu(B) > 0$ and

$$\dim_{\text{box}}(\partial A) < n, \quad \dim_{\text{box}}(\partial B) < n,$$

then A and B are equidecomposable by translations.

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Here the (upper) box dimension

$$\dim_{\text{box}}(S) = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

where $N(\varepsilon)$ is the number of cubes of side length ε needed to cover S.

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Even though the assumption on the boundary looks technical, some assumption besides the equality of measure is necessary (as shown also by Laczkovich)

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Conjecture (Gardner, 1991)

If two measurable sets $A, B \subseteq \mathbb{R}^n$ are Γ-equidecomposalbe using isometries from an amenable group Γ, then they are Γ-equidecomposable using measurable pieces.

Theorem (Grabowski, Máthé, Pikhurko, 2016)

If $A, B \subseteq \mathbb{R}^n$ are bounded, measurable such that $\mu(A) = \mu(B) > 0$ and

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then A and B are equidecomposable by translations using measurable pieces.

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Theorem (ZF) (Marks, Unger, 2017)

If $A, B \subseteq \mathbb{R}^n$ are bounded, Borel such that $\mu(A) = \mu(B) > 0$ and

$$\dim_{\text{box}}(\partial A) < n, \quad \dim_{\text{box}}(\partial B) < n,$$

then A and B are equidecomposable by translations using Borel pieces.

Action

Laczkovich constructs an action of $\mathbb{Z}^d$ on the torus $\mathbb{T}^n$ for large d, choosing $u_1, \ldots, u_d \in \mathbb{T}^n$ by

$$(k_1, \ldots, k_d) \cdot x = x + k_1 u_1 + \ldots k_d u_d$$

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Cubes

For such a free action u, the orbits look like copies of the $\mathbb{Z}^d$ and we look at finite fragments of the orbits of the form

$$F_N^u(x) = [0, N]^d \cdot x$$

Definition (discrepancy)

Given an action $\Gamma \curvearrowright (X, \mu)$, a subset $A \subseteq X$ and a finite subset F of an orbit, the **discrepancy** is defined as

$$D(F, A) = |\frac{|F \cap A|}{|F|} - \mu(A)|$$

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Discrepancy measures how well a subset A is equidistributed on the orbits.

Theorem (Laczkovich)

Let $A \subseteq \mathbb{T}^n$ be measurable such that

$$\mu(A) > 0, \quad \dim_{\text{box}}(\partial A) < n$$

and let

$$d > \frac{2n}{n - \dim_{\text{box}}(\partial A)}.$$

For almost all $u \in (\mathbb{T}^n)^d$ there exists $\varepsilon > 0$ and $M > 0$ such that for all x and all N we have

$$D(F_N^u(x), A) \leq \frac{M}{N^{1+\varepsilon}}.$$

The $\varepsilon > 0$ is crucial in both proofs of Grabowski–Máthé–Pikhurko and Marks–Unger.

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Note

Some discrepancy estimates are natural as the size of the boundary of $[0, N]^d$ relative to its size is of the form

$$\frac{2d}{N}.$$

Definition (equidistrubution)

A set $A \subseteq X$ is **equidistributed** with respect to an action $\mathbb{Z}^d \curvearrowright X$ if there exists $M > 0$ such that for μ-a.e. $x \in X$, for all N we have

$$D(F_N(x), A) \leq \frac{M}{N}$$

Note that if $\Gamma \curvearrowright X$ is a finitely generated group action, and A, B are equidecomposable, then they must satisfy a version of the Hall marriage theorem

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Definition (Hall condition)

Suppose $\Gamma \curvearrowright X$ is a finitely generated group action and $A, B \subseteq X$. The pair A, B satisfies the **Hall condition** if for every (μ-a.e.) $x \in X$ and every finite subset F of the orbit of x we have

$$|A \cap F| \leq |B \cap \mathrm{ball}(F)|, \quad |B \cap F| \leq |A \cap \mathrm{ball}(F)|.$$

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$$|A \cap F| \leq |B \cap \mathrm{ball}(F)|, \quad |B \cap F| \leq |A \cap \mathrm{ball}(F)|.$$

Here, $\mathrm{ball}(F)$ means the ball in the Cayley graph metric on the orbit. In general, this definition depends on the set of generators and we say that A, B satisfy the Hall condition is the above is **true for some set of generators**.

Fact

If A, B are equidecomposable, then A, B satisfy the Hall condition

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Proof

Suppose $\gamma_1, \ldots, \gamma_n$ are used in the decomposition. Add them as generators and then the equidecomposition is a perfect matching in the Cayley graph.

Question (Miller, 1996)

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If a measurable version of the Hall marriage theorem were true, then any two equidecomposable sets would be equidecomposable with measurable pieces...

Theorem (Marks–Unger)

Let G be a locally finite bipartite Borel graph with Borel bipartition B_0, B_1. Suppose that for some $\varepsilon > 0$ we have that for every finite set F contained in B_0 or B_1 we have

$$|F| \leq (1+\varepsilon)|\mathrm{ball}(F)|.$$

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Then there exists a Baire measurable perfect matching in G.

Note that ε appears both in the above result and in the circle squaring results...

Theorem (Cieśla–S.)

Suppose Γ is an abelian group and $\Gamma \curvearrowright (X, \mu)$ is a free pmp action. Suppose $A, B \subseteq X$ are measurable, equidistributed and $\mu(A) = \mu(B) > 0$. TFAE

(i) A, B are Γ-equidecomposable μ-a.e.

(ii) A, B satisfy the Hall condition μ-a.e.

(iii) A, B are Γ-equidecomposable μ-a.e. using μ-measurable pieces.

Corollary

Suppose Γ is an infinite f.g. abelian group and $\Gamma \curvearrowright (X, \mu)$ is a free pmp action. Suppose $A, B \subseteq X$ are measurable, equidistributed and $\mu(A) = \mu(B) > 0$.

Corollary

Suppose Γ is an infinite f.g. abelian group and $\Gamma \curvearrowright (X, \mu)$ is a free pmp action. Suppose $A, B \subseteq X$ are measurable, equidistributed and $\mu(A) = \mu(B) > 0$.

If A, B are equidecomposable, then A, B are equidecomposable using μ-measurable pieces.

This generalizes the measurable circle squaring by Grabowski, Máthé and Pikhurko

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The proof of corollary uses the following lemma.

Lemma

If A, B are equidecomposable and μ-a.e. equidecomposable using measurable pieces, then A, B are equidecomposable using measurable pieces.

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Proof

Suppose

$$A_1, \ldots, A_n, \quad B_1, \ldots, B_n,$$

with $\gamma_i A_i = B_i$ witness that A, B are equidecomposable and

$$A_1^*, \ldots, A_m^*, \quad B_1^*, \ldots, B_m^*$$

are measurable with $\delta_j A_j^* = B_j^*$ witness that A, B are μ-a.e. equidecomposable. That means that $A \setminus \bigcup_i A_i^*$ and $B \setminus \bigcup_i B_I^*$ have measure zero.

Proof

Let N be a measure zero set containing both the $A \setminus \bigcup_i A_i^*$ and $B \setminus \bigcup_i B_I^*$ and Γ-invariant. Then note that

$$\gamma_i(A_i \cap N) = B_i \cap N$$

and

$$\delta_j(A_i^* \setminus N) = B_i^* \setminus N$$

so

$$A_1 \cap N, \ldots, A_n \cap N, \quad A_1^* \setminus N, \ldots A_m^* \setminus N$$

and

$$B_1 \cap N, \ldots, B_n \cap N, \quad B_1^* \setminus N, \ldots B_m^* \setminus N$$

witness equidecomposition using measurable sets.

In the statement of Hall's theorem, only the implication from (ii) to (iii) requires a proof.

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Step 1.

We assume that CH holds

Lemma

Let $V \subseteq W$ be two models of ZF. Suppose in V we have a standard Borel space X with a Borel probability measure μ, two Borel subsets $A, B \subseteq X$ and $\Gamma \curvearrowright (X, \mu)$ is a Borel pmp action of a countable group Γ.

Lemma

Let $V \subseteq W$ be two models of ZF. Suppose in V we have a standard Borel space X with a Borel probability measure μ, two Borel subsets $A, B \subseteq X$ and $\Gamma \curvearrowright (X, \mu)$ is a Borel pmp action of a countable group Γ.

The statement that the sets A and B are Γ-equidecomposable μ-a.e. using μ-measurables pieces is absolute between V and W.

Proof

This statement an be written as

$$\exists x_1, \ldots, x_n \ \bigwedge_{i \leq n} \mathsf{BorelCode}(x_i)) \wedge \bigwedge_{i \neq j} x_i^{\#} \cap x_j^{\#} = \emptyset$$

$$\wedge\ \forall^{\mu} x\ (x \in A \leftrightarrow \bigvee_{i=1}^{n} x \in x_i^{\#}) \wedge\ \forall^{\mu} x\ (x \in B \leftrightarrow \bigvee_{i=1}^{n} x \in \gamma_i x_i^{\#})$$

and thus is it $\mathbf{\Sigma}^1_2$

This allows us to prove the theorem in $L[r]$ where r is a real conding all the necessary information.

The main trick

The main trick in the proof of Hall's theorem is the use of **Mokobodzki's medial means**, which exist under the assumption of CH (and AC).

Definition

A *medial mean* is a linear functional $\mathrm{m} : \ell_\infty \to \mathbb{R}$ which is

- positive, i.e. $\mathrm{m}(f) \geq 0$ if $f \geq 0$,
- normalized, i.e. $\mathrm{m}(1_{\mathbb{N}}) = 1$
- and shift invariant, i.e. $\mathrm{m}(Sf) = \mathrm{m}(f)$ where $Sf(n+1) = f(n)$.

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Theorem (Mokobodzki)

Under CH, there exists a median mean which is universally measurable on $[0,1]^{\mathbb{N}}$.

If the Hall condition is satisfied for A and B, then for every $\epsilon > 0$ one can find a measurable perfect matching f_n on a subset $X_n \subseteq X$ of measure $\mu(X_n) > 1 - \frac{1}{2^n}$. A perfect matching is simply a $\{0, 1\}$-valued flow.

Measurable average of flows

Using a medial mean m we can define a measurable bounded real-valued flow as

$$f = \mathrm{m}((f_n)_{n=0}^{\infty}),$$

where m is a medial mean.

Step 2

Next, we follow a strategy of Marks and Unger and modify a bounded real-valued flow to get a bounded integer-valued flow.

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For this, we give a new, explicit construction which modifies the real-valued flow "box-by-box".