SRT^2_2 vs RT^2_2 in ω-models

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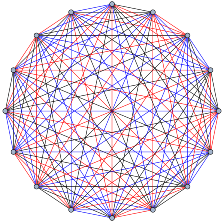

Section 1

Ramsey Theory

Motivation

It all started with this guy...

Theorem (Ramsey's theorem)

Let $n \geqslant 1$. For each coloration of $[\omega]^n$ in a finite number of color, there exists a set $X \in [\omega]^\omega$ such that each element of $[X]^n$ has the same color ($[X]^n$ is said to be monochromatic).

Motivation

Ramsey Theory

A general question

Suppose we have some mathematical structure that is then cut into finitely many pieces. How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property ?

Examples :

1. Van der Waerden's theorem
2. Hindman's theorem
3. ...

Motivation

Example (Van der Waerden's theorem)

For any given c and n, there is a number $w(c, n)$, such that if $w(c, n)$ consecutive numbers are colored with c different colors, then it must contain an arithmetic progression of length n whose elements all have the same color.

We know that :

$$w(c, n) \leqslant 2^{2^{c^{2^{2^{n+9}}}}}$$

Example (Hindmam's theorem)

If we color the natural numbers with finitely many colors, there must exists a monochromatic infinite set closed by finite sums.

Partition regularity

Theorems in Ramsey theory often assert, in their stronger form, that certain classes are *partition regular* :

Definition (Partition regularity)

A *partition regular* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

1. $\mathcal{L}$ is not empty
2. If $X \in \mathcal{L}$ and $Y_0 \cup \cdots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

Partition regularity

The following classes are partition regular :

Classical combinatorial results :

1. The class of infinite sets
2. The class of sets with positive upper density
3. The class of sets containing arbitrarily long arithmetic progressions (Van der Waerden's theorem)
4. The class of sets containing an infinite set closed by finite sum (Hindman's theorem)

... and *new* type of results involving computability :

1. Given X non-computable, the class sets containing an infinite set which does not compute X (Dzhafarov and Jockusch)

Ramsey's theorem and reverse mathematics

Theorem (Dzhafarov and Jockusch)

Given X non-computable, Given $A^0 \cup A^1 = \omega$, there exists $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G does not compute X.

This theorem comes from Reverse mathematics :

What is the computational strength of Ramsey's theorem ?

that is, given a computable coloring of say $[\omega]^2$, must all monochromatic sets have a specific computational power ?

Theorem (Seetapun)

For any non-computable set X and any computable coloring of $[\omega]^2$, there is an infinite monochromatic set which does not compute X.

Theorem (Jockusch)

There exists a computable coloring of $[\omega]^3$, every solution of which computes $\emptyset'$.

Background of RT^2_2 vs SRT^2_2

Modern approach of Seetapun's theorem (Cholak, Jockusch, Slaman) :

Definition

A set C is $\{R_n\}_{n\in\omega}$-*cohesive* if $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$ for every n.

Definition

A coloring $c : \omega^2 \to \{0,1\}$ is *stable* if $\forall x \ \lim_{y\in C} c(x,y)$ exists.

1. Given a computable coloring $c : \omega^2 \to \{0,1\}$, let $R_n = \{y \ : \ c(n,y) = 0\}$. Let C be $\{R_n\}_{n\in\omega}$-cohesive. Then c restricted to C is stable.
2. Let c be a stable coloring. Let A_c be the $\Delta^0_2(c)$ set defined as $A_c(x) = \lim_y c(x,y)$. An infinite subset of A_c or of $\overline{A_c}$ can be used to compute a solution to c.

$\to$ Find a cohesive set C (cohesive for the recursive sets) which does not compute X and use Dzhafarov and Jockusch relative to C with $A_{c\upharpoonright C}$.

Background of RT^2_2 vs SRT^2_2

Definition

RT^2_2 : Any coloring $c : \omega^2 \to \{0, 1\}$ admits an infinite homogeneous set.

The key idea of Cholak, Jockusch and Slaman is to split RT^2_2 into simpler principles (original motivation was to find a low_2 solution to RT^2_2) :

Definition

COH : For any sequence of sets $\{R_n\}_{n\in\omega}$ there is an $\{R_n\}_{n\in\omega}$-cohesive set.

Definition

SRT^2_2 : Any stable coloring admits a monochromatic set.

$\leftrightarrow$ (over RCA_0)

D^2_2 : For any Δ^0_2 set A, there is a set $X \in [A]^\omega \cup [\overline{A}]^\omega$.

We have that RT^2_2 is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$ over RCA_0.

The question

Theorem (Cholak, Jockusch and Slaman)

$\mathrm{RT}^2_2 \leftrightarrow_{\mathrm{RCA}_0} \mathrm{STR}^2_2 + \mathrm{COH}$.

Theorem (Hirschfeldt, Jockusch, Kjoss-Hanssen, Lempp and Slaman)

RT^2_2 *is strictly stronger than COH over* RCA_0.

Question

Do we have that RT^2_2 *is strictly stronger than* SRT^2_2 *over* RCA_0 *?*

$\leftrightarrow$

Do we have that SRT^2_2 *implies* COH *over* RCA_0 *?*

Theorem (Chong, Slaman, Yang)

RT^2_2 *is strictly stronger than* SRT^2_2 *over* RCA_0.

The question

Theorem (Chong, Slaman, Yang)

SRT^2_2 *does not imply* COH *over* RCA_0.

Proposition

X' is $\mathrm{PA}(\varnothing')$ iff X computes a p-cohesive set : a set which is cohesive for primitive recursive sets.
$\rightarrow$ A p-cohesive set cannot be low.

The separation is done by building a non-standard models of $\mathrm{SRT}^2_2 + \mathrm{RCA}_0$ containing only sets which are low within the model. The model has to be non-standard by the following :

Theorem (Downey, Hirschfeldt, Lempp and Solomon)

There is a Δ^0_2 set A with no infinite low set in it or in its complement.

The proof of DHLS uses Σ^0_2-induction.

Our goal

Our goal

Show that for any Δ^0_2 set A, there is an infinite set G in A or in $\overline{A}$ such that G' is not $\mathrm{PA}(\emptyset')$.

If the construction relativizes (every construction does) we can build an ω-model of $\mathrm{RCA}_0 + \mathrm{D}^2_2 \equiv \mathrm{RCA}_0 + \mathrm{SRT}^2_2$ which contains no p-cohesive set and thus which is not a model of COH.

Steps to come :

1. We explain how to use Mathias forcing to build non-cohesive and non PA sets (warm up).
2. We explain how to use Mathias forcing to control the truth of Σ^0_2 statements.
3. We sketch the actual proof.

Section 2

Partition regular classes :
A simple proof of Liu's theorem

Largeness and partition regularity

Definition (Largeness)

A *largeness* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

1. $\mathcal{L}$ is upward closed : If $X \in \mathcal{L}$ and $X \subseteq Y$, then $Y \in \mathcal{L}$
2. If $Y_0 \cup \cdots \cup Y_k \supseteq \omega$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$
3. If $X \in \mathcal{L}$ then $|X| \geqslant 2$

Definition (Partition regularity)

A *partition regular* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

1. $\mathcal{L}$ is a largeness class
2. If $X \in \mathcal{L}$ and $Y_0 \cup \cdots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

Generalities

Proposition

A partition regular class $\mathcal{L}$ contains only infinite sets.

Proposition

Let $\mathcal{L}$ be a partition regular class. Then $\mathcal{L}$ is closed by finite change of its elements. Furthermore if $\mathcal{L}$ is measurable it has measure 1.

Proof sketch :

$\mathcal{L}$ contains only infinite set

$\rightarrow$ $\mathcal{L}$ is closed by finite change

$\rightarrow$ $\mathcal{L}$ has measure 0 or 1

$\rightarrow$ If $\mathcal{L}$ has measure 0, sufficiently MLR Z and $\omega - Z$ are not in $\mathcal{L}$

$\rightarrow$ But Z or $\omega - Z$ must be in $\mathcal{L}$. Contradiction.

$\rightarrow$ $\mathcal{L}$ has measure 1

Generalities

Proposition (Compactness for largeness classes)

Suppose $\{\mathcal{A}_n\}_{n\in\omega}$ is a collection of largeness classes with $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$. Thus $\bigcap_{n\in\omega} \mathcal{A}_n$ is a largeness class.

Proposition (Compactness for partition regular classes)

Suppose $\{\mathcal{L}_n\}_{n\in\omega}$ is a collection of partition regular classes with $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n$. Thus $\bigcap_{n\in\omega} \mathcal{L}_n$ is partition regular.

Proposition

Let $\mathcal{A}$ be any set. Then $\mathcal{A}$ is a largeness class iff the set

$$\mathcal{L}(\mathcal{A}) = \{X \in 2^\omega \ : \ \forall k \ \forall X_0 \cup \cdots \cup X_k \supseteq X \ \exists i \leqslant k \ X_i \in \mathcal{A}\}$$

is a partition regular subclass of $\mathcal{A}$ (in which case it is the largest).

Π^0_2 partition regular classes

Proposition

If $\mathcal{U}$ is a Σ^0_1 large class. Then $\mathcal{L}(\mathcal{U})$ is a Π^0_2 partition regular class.

Proposition

If $\mathcal{U}$ is a Σ^0_1 upward closed class. Then predicate

$$\mathcal{U} \text{ is large}$$

is Π^0_2.

Fix k, the class of element :

$$\{Y_0 \oplus \cdots \oplus Y_k \ : \ X \subseteq Y_0 \oplus \cdots \oplus Y_k \wedge \forall i < k \ \ Y_i \notin \mathcal{U}\}$$

is a $\Pi^0_1(X)$ class uniformly in X.

Canonical Π^0_2 partition regular classes

Definition

For any infinite set X we define $\mathcal{L}_X$ as the $\Pi^0_2(X)$ partition regular class of the sets that intersect X infinitely often.

Proposition

There is a Π^0_2 partition regular class $\mathcal{L}$ such that $\mathcal{L}_X \nsubseteq \mathcal{L}$ for any $X \in [\omega]^\omega$.

The set is given by

$$\mathcal{L} = \{X \ : \ \forall k \ \exists n \ \text{ s.t. } |X \upharpoonright_{n^2}| \geqslant nk\}$$

Question

Are there any other Π^0_2 partition regular classes ?

Partition genericity

Definition

Let $\mathcal{A} \subseteq \omega$ be a largeness class. We say that X is *partition generic below* $\mathcal{A}$ if for every Σ^0_1 class $\mathcal{U}$ such that $\mathcal{A} \cap \mathcal{U}$ is large, X is in $\mathcal{A} \cap \mathcal{U}$.

If X is partition generic in 2^ω we simply say that X is *partition generic*.

We have that ω is partition-generic.

Definition

We say that X is *bi-partition generic below* $\mathcal{A}$ if X and $\omega - X$ are both partition-generic below $\mathcal{A}$.

Note that every non-trivial partition regular class if of measure 1. It follows that any Kurtz-random is *bi-partition generic*.

The key lemma for partition genericity

The class of elements which are partition generic "below something" is partition regular :

Lemma

Let C be any set such that $\bigcap_{e \in C} \mathcal{U}_e$ is large (each $\mathcal{U}_e$ is Σ^0_1). Suppose X is partition generic below $\bigcap_{e \in C} \mathcal{U}_e$. Let $Y_0 \cup \dots \cup Y_k \supseteq X$. There is a Σ^0_1 class $\mathcal{V}$ such that $\mathcal{V} \cap \bigcap_{e \in C} \mathcal{U}_e$ is large and some $i \leqslant k$ such that Y_i is partition generic below $\mathcal{V} \cap \bigcap_{e \in C} \mathcal{U}_e$.

Suppose we have Σ^0_1 classes $\mathcal{V}_n \subseteq \mathcal{V}_{n-1} \subseteq \dots \subseteq \mathcal{V}_0$ with $Y_i \notin \mathcal{V}_i$ and $\mathcal{V}_i \cap \bigcap_{e \in C} \mathcal{U}_e$ large. As X is partition generic we must have $X \in \mathcal{L}(\mathcal{V}_n \cap \bigcap_{e \in C} \mathcal{U}_e)$ and then $Y_i \in \mathcal{L}(\mathcal{V}_n \cap \bigcap_{e \in C} \mathcal{U}_e)$ for some i. Contradiction.

A simple proof of Liu's theorem

Definition

Let $\mathbb{P}$ be the set of forcing conditions $(\sigma, X, \mathcal{U})$ where :

1. $\sigma \subseteq A$ with $X \cap \{0, \dots, |\sigma|\} = \emptyset$
2. $\mathcal{U}$ is a large Σ^0_1 class
3. $X \subseteq A$ is partition generic inside $\mathcal{U}$

We have $(\sigma, Y, \mathcal{U}) \leq (\tau, Z, \mathcal{V})$ if $(\sigma, Y) \leq (\tau, Z)$ and $\mathcal{U} \subseteq \mathcal{V}$.

Definition

1. $(\sigma, X, \mathcal{U}) \Vdash \exists n\ \Phi(G, n)$ if $\exists n\ \Phi(\sigma, n)$
2. $(\sigma, X, \mathcal{U}) \Vdash \forall n\ \Phi(G, n)$ if $\forall n\ \forall \tau \subseteq X\ \Phi(\sigma \cup \tau, n)$
3. $(\sigma, X, \mathcal{U})\ ?\!\vdash \exists n\ \Phi(G, n)$ if

$$\mathcal{U} \cap \{Y\ :\ \exists \tau \subseteq Y - \{0, \dots, |\sigma|\}\ \exists n\ \Phi(\sigma \cup \tau, n)\} \text{ is large}$$

A simple proof of Liu's theorem

Lemma

Suppose $\forall n\ \exists i \in \{0,1\}\ p \,?\!\vdash \Phi(G,n)\downarrow = i$. *Then there is* $q \leqslant p$ *such that* $q \Vdash \Phi(G,n)\downarrow = \Phi_n(n)$ *for some* n.

Let $p = (\sigma, X, \mathcal{U})$. Fix $k \in \omega$. Let $f : \omega \to \{0,1\}$ be the computable function which on n finds some $i \in \{0,1\}$ such that for every k-partition $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ there is $\tau \subseteq Y_i$ for some $Y_i \in \mathcal{U}$ such that $\Phi(\sigma \cup \tau, n)\downarrow = i$.

There must be some n such that $f(n) = \Phi_n(n)$. Thus for every k-partition $Y_0 \cup \cdots \cup Y_k$ there is $\tau \subseteq Y_i$ for some $Y_i \in \mathcal{U}$ such that $\exists n\ \Phi(\sigma \cup \tau, n)\downarrow = \Phi_n(n)$.

As this is true for every k the open set $\mathcal{V} = \{Y : \exists n\ \Phi(\sigma \cup \tau, n)\downarrow = \Phi_n(n)\}$ is such that $\mathcal{U} \cap \mathcal{V}$ is large. As X is partition generic in $\mathcal{U}$ we must have $X \in \mathcal{U}$ and thus some $\tau \subseteq X$ such that $\exists n\ \Phi(\sigma \cup \tau, n)\downarrow = \Phi_n(n)$.

$(\sigma \cup \tau, X - \{0, \ldots, |\sigma \cup \tau|\}, \mathcal{U} \cap \mathcal{V})$ is a valid forcing extension of p which satisfies the lemma.

A simple proof of Liu's theorem

Lemma

Suppose $\exists n\ \forall i \in \{0,1\}\ p\ ?\not\Vdash \Phi(G,n)\downarrow = i$. Then there is $q \leqslant p$ such that $q \Vdash \Phi(G,n)\uparrow$ for some n.

Let $p = (\sigma, X, \mathcal{U})$. There is $n \in \omega$ and covers $Y^0_0 \cup \cdots \cup Y^0_k \supseteq \omega$, $Y^1_0 \cup \cdots \cup Y^1_k \supseteq \omega$ such that

1. For all $Y^0_j \in \mathcal{U}$, $\forall \tau \subseteq Y^0_j$ we have $\Phi(\sigma \cup \tau, n) \neq 0$.
2. For all $Y^1_j \in \mathcal{U}$, $\forall \tau \subseteq Y^1_j$ we have $\Phi(\sigma \cup \tau, n) \neq 1$.

Let $Y_0 \cup \cdots \cup Y_l \supseteq \omega$ be a refinement of $\{Y^0_j : j < k\}$ and $\{Y^1_j : j < k\}$. Then for every $j < l$ and for all $\tau \subseteq Y_j$ we have $Y_j \in \mathcal{U}$ implies $\Phi(\sigma \cup \tau, n)\uparrow$.

There must be $j \leqslant l$ and a large Σ^0_1 class $\mathcal{V} \subseteq \mathcal{U}$ such that $X \cap Y_j$ is partition generic in $\mathcal{V}$.
$(\sigma, X \cap Y_j, \mathcal{V})$ is a forcing extension of p which satisfies the theorem.

A slight modification

Theorem (Liu, slightly enhanced)

Let $\mathcal{L}$ is a Π^0_2 large class, If A is partition generic in $\mathcal{L}$, then there is a set $G \in [A]^\omega$ such that $G \in \mathcal{L}$ and G is not PA

We simply make sure that conditions $(\sigma, X, \mathcal{U})$ are such that $\mathcal{U} \cap \mathcal{L}$ is a large class. The proof relativizes

Theorem (Liu, relativized)

If G_0 is not PA and $\mathcal{L}$ is a $\Pi^0_2(G_0)$ large class, If A is partition generic relative to G_0 below $\mathcal{L}$, then there is a set $G_1 \in [A]^\omega$ such that $G_1 \in \mathcal{L}$ and $G_0 \oplus G_1$ is not PA.

Partition generic relative to G_0 means being in every $\Sigma^0_1(G_0)$ large class.

How about a non-cohesive solution ?

Let $X_0 \cup X_1 \cup X_2 = \omega$ be three infinite computable sets. Let $A^0 \cup A^1 = \omega$ be partition generic sets. We first find $G_0 \in [A^0]^\omega$ with $G_0 \in \mathcal{L}_{X_0}$ and G_0 not PA. We now have two possibilities :

1. A^0 is partition generic relative to G_0, somewhere below $\mathcal{L}_{X_1}$.
   $\rightarrow$ We find $G_1 \in [A^0]^\omega$ with $G_1 \in \mathcal{L}_{X_1}$ and $G_0 \oplus G_1$ not PA.
2. A^1 is partition generic relative to G_0, somewhere below $\mathcal{L}_{X_1}$.
   $\rightarrow$ We find $G_1 \in [A^1]^\omega$ with $G_1 \in \mathcal{L}_{X_1}$ and $G_0 \oplus G_1$ not PA.

We start again with $G_2 \in [A^0]^\omega \cup [A^1]^\omega$ with $G_2 \in \mathcal{L}_{X_2}$ and $G_0 \oplus G_1 \oplus G_2$ not PA.

In any case we have $G_{i_0} \cup G_{i_1} \subseteq A^0$ or $G_{i_0} \cup G_{i_1} \subseteq A^1$ for $i_0 \neq i_1$ with $G_{i_0} \cup G_{i_1} \leqslant_T G_0 \oplus G_1 \oplus G_2$ not PA and $G_{i_0} \cup G_{i_1}$ not cohesive.

Forcing in product space for non-cohesive solution

Definition (Largeness in product spaces)

A *largeness* class is a collection of sets $\mathcal{L} \subseteq (2^\omega)^n$ such that :

1. $\mathcal{L}$ is upward closed on every component : If $(X_i \ : \ i < n) \in \mathcal{L}$ and $X_i \subseteq Y_i$, then $(Y_i \ : \ i < n) \in \mathcal{L}$
2. If $Y_{i,0} \cup \cdots \cup Y_{i,k} \supseteq \omega$ for $i < n$, then there is $f : n \to k$ such that $(Y_{f(i)} \ : \ i < n) \in \mathcal{L}$
3. If $(X_i \ : \ i < n) \in \mathcal{L}$ then $|X_i| \geqslant 2$ for every i

Definition (Partition regularity in product spaces)

A *partition regular* class is a collection of sets $\mathcal{L} \subseteq (2^\omega)^n$ such that :

1. $\mathcal{L}$ is a largeness class.
2. If $(X_i : i < n) \in \mathcal{L}$ and $Y_0^i \cup \cdots \cup Y_k^i \supseteq X_i$, then there is $f : n \to k$ such that $(Y_{f(i)}^i : i < n) \in \mathcal{L}$

Forcing in product space for non-cohesive solution

Let $X_0 \cup X_1 \cup X_2 \supseteq \omega$ be three infinite computable sets. Let $A^0 \cup A^1$ be any set.

We must have $(A^{i_0}, A^{i_1}, A^{i_2})$ partition generic somewhere below $\mathcal{L}_{X_0} \times \mathcal{L}_{X_1} \times \mathcal{L}_{X_2}$. Say $i_0 = i_1 = 0$. We then have that (A^0, A^0) is partition generic somewhere below $\mathcal{L}_{X_0} \times \mathcal{L}_{X_1}$.

We then use forcing condition $(\sigma, Y_0, Y_1, \mathcal{U})$ where :

1. $Y_0 \subseteq A^0$ and $Y_1 \subseteq A^0$
2. (Y_0, Y_1) is partition generic in $\mathcal{U}$
3. $\mathcal{U} \subseteq \mathcal{L}_{X_0} \times \mathcal{L}_{X_1}$ is a largeness class

Where $(\sigma, Y_0, Y_1, \mathcal{U}) \preceq (\tau, Z_0, Z_1, \mathcal{V})$ if :

1. $(\sigma, Y_0 \cup Y_1) \preceq (\sigma, Z_0 \cup Z_1)$
2. $\mathcal{U} \subseteq \mathcal{V}$

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Section 3

Controlling Σ^0_2 statements

The non-high forcing

We shall show that for any set A, there is $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that G is not high, that is, $G' \not\geq_T \emptyset''$.

Definition

Let B be non $\Delta^0_1(\emptyset')$. Let $\mathbb{P}$ be the set of forcing conditions $p = (\sigma_0, \sigma_1, X, C)$ such that :

1. $\sigma_i \subseteq A^i$
2. B is not $\Delta^0_1(\emptyset' \oplus X \oplus C)$
3. $\mathcal{U}_C = \bigcap_{e \in C} \mathcal{U}_e$ is a $\Pi^0_2\langle C \rangle$ large partition regular class
4. X is partition generic below $\mathcal{U}_C$

We write $p^{[i]}$ for the condition (σ_i, X, C). We define $(\tau_0, \tau_1, Y, D) \leq (\sigma_0, \sigma_1, X, C)$ if $(\tau_i, Y) \leq (\sigma_i, X)$ and $C \subseteq D$.

We suppose in addition that for any such forcing condition we have that $X \cap A^0$ and $X \cap A^1$ are partition generic inside $\mathcal{U}_C$.

Definition

Given a Δ_0 formula $\Phi_e(G, n, m)$ we write $\zeta(e, \sigma, n)$ for an index of the following upward closed Σ^0_1 class :

$$\{X \ : \ \exists \tau \subseteq X - \{0, \dots, |\sigma|\} \ \exists m \ \neg\Phi_e(\sigma \cup \tau, n, m)\}$$

Definition

Let $p = (\sigma_0, \sigma_1, X, C)$. Given a Δ_0 formula $\Phi_e(G, n, m)$ we define :

1. $p^{[i]} \Vdash \exists n \ \forall m \ \Phi_e(G, n, m)$ if $(\sigma_i, X) \Vdash \forall m \ \Phi_e(G, n, m)$ for some n
2. $p^{[i]} \Vdash \forall n \ \exists m \ \neg\Phi_e(G, n, m)$ if for all n for all $\tau \subseteq X$ we have $\zeta(e, \sigma_i \cup \tau, n) \in C$

Definition

Let $\mathcal{F} \subseteq \mathbb{P}$ be a filter, so we have conditions $(\sigma^0_0, \sigma^0_1, \dots) \geq (\sigma^1_0, \sigma^1_1, \dots) \geq (\sigma^2_0, \sigma^2_1, \dots) \geq$ in $\mathbb{P}$. We write $G^i_{\mathcal{F}}$ for the sequence $\sigma^0_i \preceq \sigma^1_i \preceq \sigma^2_i \preceq \dots$.

Lemma (Truth lemma for Σ^0_2)

Let $p = (\sigma_0, \sigma_1, X, C)$. Suppose $p^{[i]} \Vdash \exists n\ \forall m\ \Phi_e(G, n, m)$ If $\mathcal{F}$ is generic enough with $p \in \mathcal{F}$ we have $\exists n\ \forall m\ \Phi_e(G^i_{\mathcal{F}}, n, m)$

For some n, for all $\tau \subseteq X$ and all m we have $\Phi_e(\sigma_i \cup \tau, n, m)$. Then clearly $\exists n\ \forall m\ \Phi_e(G^i_{\mathcal{F}}, n, m)$.

Lemma (Extension lemma for Π^0_2)

Let $p = (\sigma_0, \sigma_1, X, C)$. Suppose $p^{[i]} \Vdash \forall n\ \exists m\ \neg\Phi_e(G, n, m)$. Let $q \leq p$. with $q = (\tau_0, \tau_1, Y, D)$. Then $q^{[i]} \Vdash \forall n\ \exists m\ \neg\Phi_e(G, n, m)$

For every $\tau \subseteq X$ and every n we have $\zeta(\sigma_i \cup \tau, n) \in C \subseteq D$. We have $\tau_i = \sigma_i \cup \tau$ for some $\tau \subseteq X$. Then also for every $\rho \subseteq Y \subseteq X$ we have $\zeta(\sigma_i \cup \tau, n) \in D$.

Lemma (Truth lemma for Π^0_2)

Let $p = (\sigma_0, \sigma_1, X, C)$. Suppose $p^{[i]} \Vdash \forall n \, \exists m \, \neg\Phi_e(G, n, m)$ If $\mathcal{F}$ is generic enough with $p \in \mathcal{F}$ we have $\forall n \, \exists m \, \neg\Phi_e(G^i_{\mathcal{F}}, n, m)$

We shall show that for every n the set

$$\{(\tau_0, \tau_1, Y, D) \; : \; (\tau_i, Y) \Vdash \exists m \, \neg\Phi_e(G, n, m)\}$$

is dense below p. If $\mathcal{F}$ is generic enough it has a condition in each of these dense set and then $\forall n \, \exists m \, \neg\Phi_e(G^i_{\mathcal{F}}, n, m)$

Fix x. Let $q \leq p$ with $q = (\tau_0, \tau_1, Y, D)$. Then $q^{[i]} \Vdash \forall n \, \exists m \, \neg\Phi_e(G, n, m)$. It follows that $\zeta(e, \tau_i, n) \in D$. Also $X \cap A^i \in \mathcal{U}_D$. It follows that there exists $\rho \subseteq X \cap A^i$ such that $\exists m \, \neg\Phi_e(\tau_i \cup \rho, n, m)$. $(\tau_{1-i}, \tau_i \cup \rho, X - \{0, \ldots, |\tau_i \cup \rho|\}, D)$ is a valid extension of q for which $(\tau_i \cup \rho, X - \{0, \ldots, |\tau_i \cup \rho|\}) \Vdash \exists m \, \neg\Phi_e(G, n, m)$.

Definition (The forcing question)

Let $p = (\sigma_0, \sigma_1, X, C)$. We define $p \,?\!\vdash \exists n\, \forall m\, \Phi_{e_0}(G, n, m) \vee \exists n\, \forall m\, \Phi_{e_1}(G, n, m)$ iff

$$\forall Z^0 \cup Z^1 \supseteq X \quad \bigcap_{\tau \subseteq Z^0, n \in \omega} \mathcal{U}_{\zeta(e_0, \sigma_0 \cup \tau, n)} \cap \bigcap_{\tau \subseteq Z^1, n \in \omega} \mathcal{U}_{\zeta(e_1, \sigma_1 \cup \tau, n)} \cap \mathcal{U}_C$$

is not large

Proposition

The forcing question is $\Sigma^0_1(X \oplus C \oplus \emptyset')$

We have $p \,?\!\vdash \exists n\, \forall m\, \Phi_{e_0}(G, n, m) \vee \exists n\, \forall m\, \Phi_{e_1}(G, n, m)$ iff for every $Z^0 \cup Z^1 \supseteq X$ there exists a finite set $F \subseteq C$ togther with $\tau_0^0, \dots, \tau_0^k \subseteq Z^0$ with $\tau_1^0, \dots, \tau_1^k \subseteq Z^1$ and $n_1, \dots, n_k$ such that the Σ^0_1 class :

$$\bigcap_{\tau_0^i, n_i} \mathcal{U}_{\zeta(e_0, \sigma_i \cup \tau_0^i, n_i)} \cap \bigcap_{\tau_1^i, n_i} \mathcal{U}_{\zeta(e_1, \sigma_i \cup \tau_1^i, n_i)} \cap \mathcal{U}_F$$

is not large

Lemma

Suppose $p \,?\!\Vdash \exists n\, \forall m\, \Phi_{e_0}(G, n, m) \vee \exists n\, \forall m\, \Phi_{e_1}(G, n, m)$ Then there exists $q \leqslant p$ and $i \in \{0, 1\}$ such that $q^i \Vdash \exists n\, \forall m\, \Phi_{e_i}(G, n, m)$

We have for every $Z^0 \cup Z^1 \supseteq X$ that there exists a finite set $F \subseteq C$ togther with $\tau_0^0, \dots, \tau_0^k \subseteq Z^0$ with $\tau_1^0, \dots, \tau_1^k \subseteq Z^1$ and $n_1, \dots, n_k$ such that the Σ^0_1 class :

$$\mathcal{V} = \bigcap_{\tau_0^i, n_i} \mathcal{U}_{\zeta(e_0, \sigma_0 \cup \tau_0^i, n_i)} \cap \bigcap_{\tau_1^i, n_i} \mathcal{U}_{\zeta(e_1, \sigma_1 \cup \tau_1^i, n_i)} \cap \mathcal{U}_F$$

is not large.
Take $Z^0 = A^0$ and $Z^1 = A^1$. There must be a cover $Y_0 \cup \dots \cup Y_k \supseteq \omega$ such that $Y_j \notin \mathcal{V}$ for $j \leqslant k$. We can furthermore assume $Y_0 \cup \dots \cup Y_k \leqslant_T \emptyset'$. There must be $j \leqslant k$ such that $Y_j \cap X$ is partition generic inside $\mathcal{U}_D$ for some $D = C \cup \{e\}$. In particular $Y_j \cap X \in \mathcal{U}_F$ and then there must be $i < 2$ with $\tau_i^l \subseteq A^i$ and n_l such that $Y_j \cap X \notin \mathcal{U}_{\zeta(e_i, \sigma_i \cup \tau_i^l, n_l)}$. Thus $\forall \rho \subseteq Y_j \cap X$ we have $\forall m\, \Phi_{e_i}(\sigma^i \cup \tau_i^l \cup \rho, n_l)$. It follows that $(\sigma_{1-i}, \sigma^i \cup \tau_i^l, Y_j \cap X, D)$ is a valid extension which satisfies the lemma.

Proposition

Suppose $p \,?\!\not\Vdash \exists n\ \forall m\ \Phi_{e_0}(G,n,m) \vee \exists n\ \forall m\ \Phi_{e_1}(G,n,m)$ *Then there exists* $q \leqslant p$ *and* $i \in \{0,1\}$ *such that* $q^{[i]} \Vdash \forall n\ \exists m\ \neg\Phi_{e_i}(G,n,m)$

The class $Z^0 \cup Z^1 \supseteq X$ such that

$$\bigcap_{\tau \subseteq Z^0, n \in \omega} \mathcal{U}_{\zeta(e_0, \sigma_0 \cup \tau, n)} \cap \bigcap_{\tau \subseteq Z^1, n \in \omega} \mathcal{U}_{\zeta(e_1, \sigma_1 \cup \tau, n)} \cap \mathcal{U}_C$$

is large, is a non-empty $\Pi^0_1(X \oplus C \oplus \emptyset')$ class. Take $Z^0 \cup Z^1$ such that B is not $\Delta^0_1(Z^0 \oplus Z^1 \oplus C \oplus \emptyset')$. Let D be C together with $\zeta(e_0, \sigma_0 \cup \tau, n)$ for every $\tau \subseteq Z_0$ and every n and with $\zeta(e_1, \sigma_1 \cup \tau, n)$ for every $\tau \subseteq Z_1$ and every n. We have that $\mathcal{U}_D$ is large. As X is partition generic inside $\mathcal{U}_C$ we must have that $Z_i \cap X$ is partition generic inside $\mathcal{U}_E$ for some $E = D \cup \{e\}$ and some $i \in \{0,1\}$.
We have that $(\sigma_0, \sigma_1, Z_i \cap X, E)$ is a valid extension of p which satisfies the lemma.

Cone avoidance

Given $p \in \mathbb{P}$. Given $\Phi_{e_0}(G, x, n, m)$ and $\Phi_{e_1}(G, x, n, m)$ the set

$$S = \{x \ : \ p\, ?\!\vdash \exists n\ \forall m\ \Phi_{e_0}(G, x, n, m) \vee \exists n\ \forall m\ \Phi_{e_1}(G, x, n, m)\}$$

is $\Sigma^0_1(p)$. As B is not $\Sigma^0_1(p)$ we have $B \neq S$. Find $q \leqslant p$ such that for some $i \in \{0, 1\}$:

1. $q^{[i]} \Vdash \exists n\ \forall m\ \Phi_{e_i}(G, x, n, m)$ for $x \notin B$
2. or $q^{[i]} \Vdash \forall n\ \exists m\ \neg\Phi_{e_i}(G, x, n, m)$ for $x \in B$.

Then by a pairing argument we must have :

1. $G^0 \subseteq A^0$ so that B is not $\Sigma^0_1((G^0)')$
2. or $G^1 \subseteq A^1$ so that B is not $\Sigma^0_1((G^1)')$.

Non-high forcing : The degenerate case

Suppose now that we encounter $p = (\sigma_0, \sigma_1, X, C)$ such that $A^i \cap X$ is not partition generic in $\mathcal{U}_C$ for $i \in \{0, 1\}$. Say $i = 1$. Then there must be a large Σ^0_1 class $\mathcal{U}$ such that X is partition generic in $\mathcal{U}$ and $X \cap A^1 \notin \mathcal{U}$.
We use forcing conditions (σ, Y, C) with :

1. $\sigma \subseteq A^0$
2. $Y \subseteq X$
3. $\mathcal{U}_C \subseteq \mathcal{U}$

The forcing question becomes

Definition (The forcing question)

Let $p = (\sigma, Y, C)$. We define $p \mathrel{?\vdash} \exists n\, \forall m\, \Phi_e(G, n, m)$ iff

$$\forall Z^0 \cup Z^1 \supseteq Y\ \exists i \in \{0, 1\}\ Y \cap Z^i \in \mathcal{U} \wedge \bigcap_{\tau \subseteq Z^i, n \in \omega} \mathcal{U}_{\zeta(e, \sigma \cup \tau, n)} \cap \mathcal{U}_C$$

is not large

More cone avoiding forcing

The non-high forcing cannot be extended in a straightforward way to control the truth of Σ^0_n statement for $n > 2$.

For $n = 3$ on would need to use large classes for the truth of Σ^0_1 statements, together with large classes for the truth of Σ^0_2 statements : the two could be incompatible.
We can however bring non-trivial modification in order to show the following :

Theorem (M., Patey)

If B is not $\Delta^0_1(\emptyset^{(\alpha)})$ for $\alpha < \omega_1^{ck}$, any set A sufficiently partition generic (below something) contains an infinite subset G such that B is not $\Delta^0_1(G^{(\alpha)})$.

Theorem (M., Patey)

If B is not Δ^1_1, any set A sufficiently partition generic (below something) contains an infinite subset G such that B is not $\Delta^1_1(G)$ (with in particular $\omega_1^G = \omega_1^{ck}$).

Section 4

Forcing non-cohesive

How to attack the problem ?

We now suppose that $A^0 \cup A^1 \supseteq \omega$ is Δ^0_2. Some obstacles prevent us from considering arbitrary sets A : essentially the problem is that members of a $\Pi^0_1(\emptyset')$ class might all be $\mathrm{PA}(\emptyset')$.

The formula $\Phi_e(G', n) \downarrow = i$ is a Σ^0_2 formula $\exists n \, \forall m \; \Phi_{f(e,i)}(G, n, m)$. Having A Δ^0_2 we can ask the following $\Sigma^0_1(\emptyset')$ question : Is the set

$$\bigcap_{\tau \subseteq A, n \in \omega} \mathcal{U}_{\zeta(f(e,i), \sigma \cup \tau, n)}$$

not a largeness class ?

If the answer is no for both $i = 0$ and $i = 1$ we have two largeness classes C_0 and C_1. Each class C_i can be used to force $\Phi_e(G', n) \neq i$. The problem is the following : The class $\mathcal{U}_{C_0} \cap \mathcal{U}_{C_1}$ need not to be large. So we instead work with the product class $\mathcal{U}_{C_0} \times \mathcal{U}_{C_1}$, so the generic can take elements in both $\mathcal{U}_{C_0}$ and $\mathcal{U}_{C_1}$.

How to attack the problem ?

Suppose we now work within $\mathcal{U}_{C_0} \times \mathcal{U}_{C_1}$. The next question to ask is of the form :

$$\forall k\ \forall X^0_0 \cup \dots \cup X^0_k \supseteq \omega\ \forall X^1_0 \cup \dots \cup X^1_k \supseteq \omega\ \exists i_0, i_1 < k \\ (X^0_{i_0}, X^1_{i_1}) \in \mathcal{U}_{C_0} \times \mathcal{U}_{C_1} \wedge \exists \tau \subseteq X^0_{i_0} \cup X^1_{i_1} \text{ s.t. } \dots$$

If the answer is yes we continue with a large class $\mathcal{L} \subseteq \mathcal{U}_{C_0} \times \mathcal{U}_{C_1}$.

Problem : (A^0, A^0) or (A^1, A^1) need to be partition generic in $\mathcal{U}_{C_0} \times \mathcal{U}_{C_1}$ and then it may not belong to $\mathcal{L}$. It may be that $(A^0, A^1) \in \mathcal{L}$ or $(A^1, A^0) \in \mathcal{L}$. We need a product of three large classes, so that being any two of them is enough.

valuations

Definition (Liu)

1. A valuation is a partial finite function $v \subseteq \omega \to \{0, 1\}$.
2. A valuation v is $\emptyset'$-correct if $\forall n \in \text{dom}\, v \;\; v(n) = \Phi_n(\emptyset', n)$.
3. Two valuations v_1, v_2 are incompatible if $v_1(n) \neq v_2(n)$ for some $n \in \text{dom}\, v_1 \cap \text{dom}\, v_2$

Theorem (Liu)

Let V be a $\emptyset'$-c.e. set of valuation. Either V contains a $\emptyset'$-correct valuation or for any k there are k pairwise incompatible valuations outside of V.

Using valuations

Given a valuation v let $f(v)$ be the such that

$$\exists n\ \forall m\ \Phi_{f(v)}(G, n, m) \equiv \exists n \in \operatorname{dom} v\ \Phi(G', n) \downarrow = v(n)$$

Let

$$V = \{v\ :\ p\, ?\!\vdash \exists n \in \operatorname{dom} v\ \Phi_n(G', n) \downarrow = v(n)\}$$

1. Either V contains a correct valuation v in which case we find an extension $q \leqslant p$ such that $q \Vdash \Phi_e(G', n) \downarrow = \Phi_n(\emptyset', n)$
2. Or we find 3 pairwise incompatible valuations v_1, v_2, v_3 such that for $j \leqslant 3$ the set :

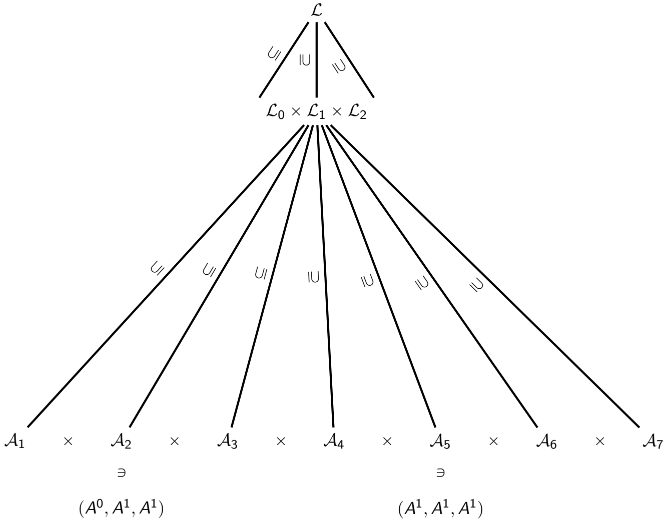
$$\mathcal{L}_j = \bigcap_{\tau \subseteq A^i, n} \mathcal{U}_{\zeta(f(v_j)), \sigma \cup \tau, n)}$$

   is large.

We start three possible generics from there

1. $G^i_{\{0,1\}} \in \mathcal{L}_0 \times \mathcal{L}_1$ with $G^i_{\{0,1\}} \subseteq A^i$
2. $G^i_{\{1,2\}} \in \mathcal{L}_1 \times \mathcal{L}_2$ with $G^i_{\{1,2\}} \subseteq A^i$
3. $G^i_{\{0,2\}} \in \mathcal{L}_0 \times \mathcal{L}_2$ with $G^i_{\{0,2\}} \subseteq A^i$

Evolution of largeness classes



When forcing our second Π^0_2 statement we need 7 pairwise incompatible valuations to end up in a large subclass of $(2^\omega)^{21}$.

The $\mathbb{P}$-forcing

Definition

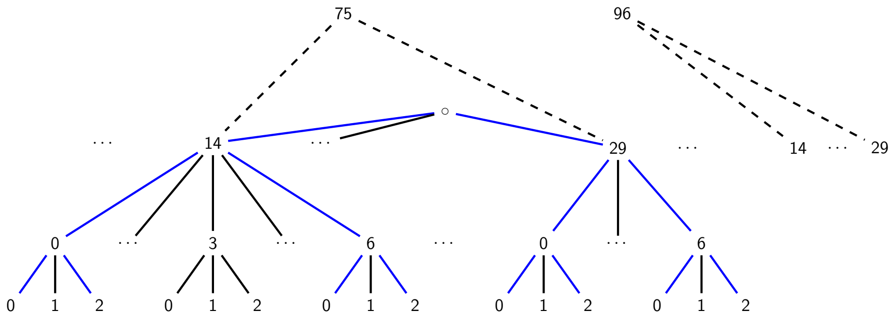
1. Let $u_0 = 1$. Let $u_{n+1} = \binom{2u_n+1}{2}$.
2. Let I_n be the set of strings σ of length n such that $\sigma(n-m) < u_m$ for $m < n$ (see picture on next slide).
3. We write $I \lhd I_n$ if I is the set of leaf of a binary subtree of I_n (where I_n is seen as a finite tree), such that for every branching node σ of I, the left subtree of σ equals the right subtree of σ.

Definition

Let $\mathbb{P}$ be the set of conditions $\langle(\sigma^I_0, \sigma^I_1 \ : \ I \lhd I_n), (X_\tau \ : \ \tau \in I_n), \mathcal{L}\rangle$ for some n.

1. $\sigma^I_i \subseteq A_i$
2. $\mathcal{L} \subseteq (2^\omega)^{|I_n|}$ is a large class
3. $(X_\tau \ : \ \tau \in I_n)$ is partition generic in $\mathcal{L}$

Illustration of $I \lhd I_n$



The blue part is some $I \lhd I_3$. The set I_4 is given by the tree $\{a\sigma \ : \ \sigma \in I_3, a \leqslant u_4\}$. The dashed part correspond to some potential extension $J \lhd I_4$ of I (where the tree below 75 equals the tree below 96).

The $\mathbb{Q}$-forcing

Definition

Let $\mathbb{Q}$ be the set of conditions $\langle \sigma_0, \sigma_1, (X_\tau : \tau \in I), \mathcal{L} \rangle$ for some $I \lhd I_n$ such that :

1. $\sigma_i \subseteq A_i$
2. $\mathcal{L} \subseteq (2^\omega)^{|I|}$ is a large class
3. $(X_\tau : \tau \in I)$ is partition generic in $\mathcal{L}$

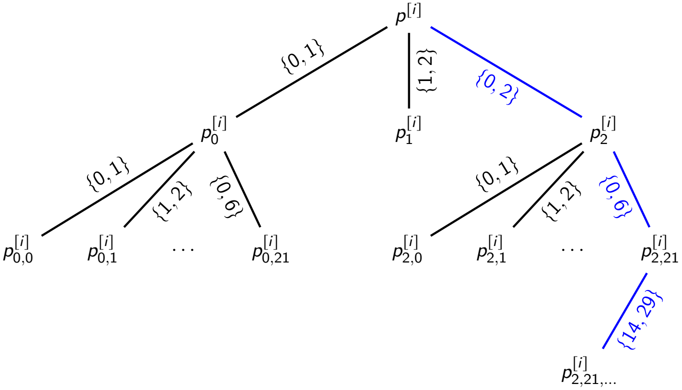
A $\mathbb{Q}$ condition p is i-valid if $(X_\tau \cap A^i : \tau \in I) \in \mathcal{L}$

Let $p \in \mathbb{P}$ with $p = \langle (\sigma_0^I, \sigma_1^I : I \lhd I_n), (X_\tau : \tau \in I_n), \mathcal{L} \rangle$ for some n. Let $I \lhd I_n$. Then p_I is the $\mathbb{Q}$ condition defined by

$$p_I = \langle \sigma_0^I, \sigma_1^I, (X_i : \tau \in I), \pi^I(\mathcal{L}) \rangle$$

where $\pi^I(\mathcal{L})$ is the projection of $\mathcal{L}$ on the components corresponding to I.

The tree of $\mathbb{Q}$-condition



The combinatorics make sure that the tree of $\mathbb{Q}$ conditions always have a valid branch of length n for every n. The blue branch correspond to the blue $I \lhd I_n$ from two slides ago.

The forcing question

Let $(\sigma^0_{0,1}, \sigma^1_{0,1}, \sigma^0_{1,2}, \sigma^1_{1,2}, \sigma^0_{0,2}, \sigma^1_{0,2}, (X_0, X_1, X_2), \mathcal{L})$ be a $\mathbb{P}$-condition. Let $\zeta(e, \sigma_{0,1}, \sigma_{1,2}, \sigma_{0,2}, n)$ be a code for the open set

$$\left\{ \begin{array}{lcl} (Y_0, Y_1, Y_2) & : & \exists \tau_{0,1} \subseteq Y_0 \cup Y_1 \ \exists m \ \Phi_e(\sigma_{0,1} \cup \tau_{0,1}, n, m) \wedge \\ & & \exists \tau_{1,2} \subseteq Y_1 \cup Y_2 \ \exists m \ \Phi_e(\sigma_{1,2} \cup \tau_{1,2}, n, m) \wedge \\ & & \exists \tau_{0,2} \subseteq Y_0 \cup Y_2 \ \exists m \ \Phi_e(\sigma_{0,2} \cup \tau_{0,2}, n, m) \end{array} \right\}$$

Given a formula $\exists n \ \forall m \ \Phi_e(G, n, m)$ the question $p^{[i]} \ ?{\vdash} \ \exists n \ \forall m \ \Phi_e(G, n, m)$ is defined by : Is the class

$$\mathcal{L} \cap \bigcap_{\tau \subseteq A^i, n \in \omega} \mathcal{U}_{\zeta(e, \sigma^i_{0,1} \cup \tau, \sigma^i_{1,2} \cup \tau, \sigma^i_{0,2} \cup \tau, n)}$$

not a largeness class ?

Make some progress

Let

$$V = \{v \ : \ p^{[i]} ?\vdash \exists n \ \forall m \ \Phi_{f(e,v)}(G, n, m)\}$$

1. If V contains a correct valuation we can extend one branch of the tree to force the jump of our generic (along that branch) to equal $\Phi_n(n)$ for some n.
2. Otherwise there must be k pairwise incompatible valuations for k as large as we want. We take k to be $2u_n + 1$. We find k largeness subclasses of our current large class. This splits each branch of our tree with $\binom{u_{n+1}}{2}$ children. On each of them we force the jump our generic to disagree everywhere with two pairwise incompatible valuation and then to be partial.

Note that if the outcome (1) occurs, we have to ask the forcing question again, but excluding the branch on which we made some progress.