Metric Scott analysis

André Nies

IMS workshop
Higher Recursion Theory and Set Theory
2019

Metric Scott analysis: the story so far

2012 I visited the KGRC in Wien, working with Sy Friedman and Katia Fokina on various topics. In the discussions the question emerged whether the usual Scott rank of a Polish metric space is countable. We viewed such a space as a structure with distance relations $R_q(x, y)$ that $d(x, y) < q$, for fixed $q \in \mathbb{Q}^+$.

2013 With Tsankov, and then Ben Yaacov and Doucha developed a version of Scott analysis where the space is viewed as a continuous structure. We related it to the concept of Gromov-Hausdorff distance between metric spaces. This happened during a Paris visit and then at the Bonn Hausdorff Institute Programme "Universality and Homogeneity".

2014 Michal Doucha's paper in the Ann. Pure Applied Logic shows that the classical Scott rank of any Polish metric space is at most ω_1.

Metric Scott analysis: the story so far

2016
- I visited Caltech and gave a talk on this topic, and suggested to William Chan, then a PhD student of Kechris, to study possible algorithmic versions.
- Chan obtained such versions. He also noticed an error at the very end of Doucha paper.
- The question is open (again): whether the classical Scott rank of a Polish metric space is countable .

2017 Paper on the continuous version of metric Scott analysis, with Ben Yaacov, Doucha and Tsankov appears.

What is Scott analysis?

Broad definition: Scott analysis means assigning ordinal valued ranks to elements of structures, and whole structures, in order to measure complexity.

- Given a structure M, for tuples $\overline{a}, \overline{b} \in M^k$, an ordinal

$$\text{rank}(\overline{a}, \overline{b})$$

  measures the complexity of distinguishing them by their properties within M.

- If they can't be distinguished at all, i.e. $\overline{b} = f(\overline{a})$ for some automorphism f of M, we define $\text{rank}(\overline{a}, \overline{b}) = \infty$.

- We use this to measure the complexity of orbits, and of the whole structure.

The back and forth games $G^M_\alpha(\overline{a}, \overline{b})$, and Scott relations

Player 1 (also called $\forall$ player, or spoiler) challenges by providing a side and an element c.

Player 2 ($\exists$ player, duplicator) has to provide an element d on the other side that behaves similarly on the previous level.

There is an initial ordinal α. At each round, Player 1 picks an ordinal smaller than the previous one.

Definition (Scott relations)

- The ground Scott relation $r_0(\overline{a}, \overline{b})$ says that $\overline{a}, \overline{b}$ satisfy the same atomic diagram.
- For $\alpha > 0$, the Scott relation $r^M_\alpha(\overline{a}, \overline{b})$ holds if Player 2 has a winning strategy when the initial ordinal is α.
- $\mathrm{rank}(\overline{a}, \overline{b})$ is the least α so that Player 2 has no winning strategy, or ∞ if no such α exists.

Ranks of orbits, rank of the whole structure

- $\mathrm{rank}(\overline{a}, \overline{b})$ is the least α so that Player 2 has no w.s., or ∞ if no such α exists.
- $\mathrm{rank}(\overline{a})$ is complexity of the orbit of $\overline{a}$, i.e. the difficulty of distinguishing $\overline{a}$ from all $\overline{b}$ that are not automorphic to $\overline{a}$. So this is $\sup\{\mathrm{rank}(\overline{a}, \overline{b})\colon \neg\overline{a} \approx \overline{b}\}$.
- $\mathsf{SR}(M)$, the Scott rank of M, is the sup of the $\mathrm{rank}(\overline{a})$ for all tuples $\overline{a}$ (of any length). This ordinal measures the complexity of M.
- We have $\mathsf{SR}(M) < |M|^+$.

Complexity

Assume that model M is countable. Let S be the signature of M. These complexity measures closely correspond to the descriptive complexity in $L_{\omega_1,\omega}(S)$.
The quantifier rank of a formula is an ordinal given by the number of quantifiers, infinite disjunctions, conjunctions.

- $r_\alpha(\overline{a}, \overline{b})$ means that $\overline{a}$ and $\overline{b}$ agree on formulas of complexity up to α.
- $\mathrm{rank}(\overline{a})$ is (about) the complexity of a formula defining the orbit of $\overline{a}$.
- $\mathsf{SR}(M)$ is (about) the complexity of a Scott sentence for M, i.e. a description among the countable structures.
- It is also (about) the Borel complexity of $\{N \colon N \cong M\}$ (assumings each model has domain ω).

Montalbán, A robuster Scott rank, PAMS 2015, gives an alternative definition of Scott rank where the last three become equalities.

Scott relations approximate the isomorphism relation

- The isomorphism relation between countable structures is not Borel in general.
- We can extend the games to pairs of countable structures M, N and tuples $\overline{a} \in M^k, \overline{b} \in N^k$.
- The $r_\alpha^{M,N}$ are Borel relations approximating isomorphism.
- If α is an ordinal so that all relations have stabilized (i.e. $r_{\alpha,n}^{M,N} = r_{\alpha+1,n}^{M,N}$ for each n) then

  $$r_\alpha^{M,N}(\overline{a}, \overline{b}) \Leftrightarrow f(\overline{a}) = \overline{b} \text{ for some isomorphism } f : M \cong N.$$

What is metric Scott analysis?

We are given a Polish metric space (M, d) (i.e. complete and separable). More generally M could be a Polish metric structure, which also has a bunch of closed relations.

Classical metric Scott analysis: We view (M, d) as a structure for the signature

$$\{R_q \colon q \in \mathbb{Q}^+\},$$

where $R_q xy$ means that $d(x, y) < q$.

Continuous metric Scott analysis:
The r^M_α are now uniformly continuous real-valued functions.
$r^M_\alpha(\overline{a}, \overline{b}) > \epsilon$ means: in α rounds Player 1 can win the ϵ game.

Examples of classical metric Scott ranks

Natural spaces tend to have low Scott rank. For instance,

- A Polish metric space has Scott rank 0 iff it is ultrahomogeneous. So Urysohn space $\mathbb{U}$ has Scott rank 0.
- A compact metric space X has Scott rank at most ω: if $\overline{a}, \overline{b} \in X^k$ satisfy the same existential formulas, then they are isometric in X.

Theorem (S. Friedman, Fokina, Koerwien, N., 2012)

For each $\alpha < \omega_1$, there is a countable discrete ultrametric space M of Scott rank $\alpha \cdot \omega$.

M is given as the maximal branches on a subtree of $\omega^{<\omega}$. For $\sigma \neq \tau \in M$, the distance is 2^{-k} where k is the least disagreement.

Metric Scott analysis with distance relations

(some more detail)

Back and forth games along a linear order L

For a structure M, a linear order L, and $\overline{a}, \overline{b} \in M^n$, the game $G_L^M(\overline{a}, \overline{b})$ is played as follows:

- In the i-th round, Player 1 chooses a $z_i \in L$ with $z_i <_L z_{i-1}$ when $i > 0$, and either chooses an element $a_{n+i} \in M$ or an element $b_{n+i} \in M$.
- Player 2 then chooses whichever of a_{n+i} or b_{n+i} Player 1 did not choose.

- After round i, if the map from $a_0 a_1 \dots a_{n+i}$ to $b_0 b_1 \dots b_{n+i}$ is not a partial isomorphism, then Player 1 wins.

- The game ends in a win for Player 2 after either ω many rounds, or if Player 1 has not already won but cannot choose a $z_{i+1} <_L z_i$.

Definition of ranks

- Fix a metric space M and a subset $D \subseteq M$ (later on a countable dense subset).
- Define the game $G_L^M(\overline{a}, \overline{b}, D)$ exactly as $G_L^M(\overline{a}, \overline{b})$, except that Player 1's choice of elements is restricted to D.

Definition (Ordinal ranks w.r.t. a subset D)

Let $\mathrm{rank}^M(\overline{a}, \overline{b}, D)$ to be

- the least ordinal α for which Player 2 does not have a winning strategy in $G_\alpha^M(\overline{a}, \overline{b}, D)$, or
- $\mathrm{rank}^M(\overline{a}, \overline{b}, D) = \infty$ if there is no such α.

$$\begin{aligned} \mathrm{rank}^M(\overline{a}, D) &:= \sup\{\mathrm{rank}^M(\overline{a}, \overline{b}, D) : \mathrm{rank}^M(\overline{a}, \overline{b}) < \infty\}. \\ \mathsf{SR}(M, D) &:= \sup\{\mathrm{rank}^M(\overline{a}, D) : \overline{a} \in M\}. \end{aligned}$$

Winning strategy of Player 2 for $G_L^M(\overline{a}, \overline{b})$ restricts to winning strategy for $G_L^M(\overline{a}, \overline{b}, D)$. So $\mathrm{rank}^M(\overline{a}, \overline{b}) \leq \mathrm{rank}^M(\overline{a}, \overline{b}, D)$.

Strategies and ill-foundedness of L

- Given some numbering of L and countable dense set $D \subseteq M$, a winning strategy for Player 2 is coded by a real.
- Checking that this real codes a winning strategy for Player 2 is arithmetical relative to the metric on D.

Lemma

Suppose L is ill-founded, then:
Player 2 has a winning strategy in $G^L_M(\overline{a}, \overline{b}, D) \Leftrightarrow$
there is an auto-isometry of M taking $\overline{a}$ to $\overline{b}$.

Proof of $\Rightarrow$:
The auto-isometry and its inverse only have to be defined on D to extend to the whole space, by completeness of M.
Player 1 plays along the infinite descending chain of L, choosing the next element of D and alternating sides.

Player 2 uses her strategy.

The rank of a pair of non-isometric tuples is relatively computable

By $M \restriction_D$ we denote the metric structure restricted to a fixed countable dense set D.

Theorem (Chan, 2016; N and Turetsky, 2017)

Suppose that $\operatorname{rank}^M(\overline{a}, \overline{b}, D)$ is not a computable ordinal relative to $(\overline{a}, \overline{b}, M \restriction_D)$.

Then there is an autoisometry of M taking $\overline{a}$ to $\overline{b}$.

Corollary (Doucha, 2014)

(i) $\operatorname{rank}^M(\overline{a}) \leq \omega_1$ *for each* $\overline{a}$

(ii) $\operatorname{rank}(M) \leq \omega_1$.

Proof (i):

- for each $\overline{b} \not\approx \overline{a}$, the ordinal $\operatorname{rank}^M(\overline{a}, \overline{b}, D)$ is relatively computable, and hence countable.
- $\operatorname{rank}^M(\overline{a}, \overline{b}) \leq \operatorname{rank}^M(\overline{a}, \overline{b}, D)$.

Theorem (Chan, 2016; N and Turetsky, 2017)

Suppose $\mathrm{rank}_D^M(\overline{a},\overline{b}) \geq \omega_1^{CK(\overline{a},\overline{b},M\restriction_D)}$.
Then there is an auto-isometry of M taking $\overline{a}$ to $\overline{b}$.

. Chan used admissible sets, we use Σ_1^1 bounding instead.

Proof.

- Let $\theta(e)$ state that $\Phi_e^{\overline{a},\overline{b},M\restriction_D}$ gives a total linear order L, and there is a real which codes a winning strategy for Player 2 in $G_L^M(\overline{a},\overline{b},D)$.
- θ is $\Sigma_1^1(\overline{a},\overline{b},M\restriction_D)$.
- $\theta(e)$ holds for every e with $\Phi_e^{\overline{a},\overline{b},M\restriction_D}$ well-ordered, for otherwise $\mathrm{rank}_D^M(\overline{a},\overline{b}) < \omega_1^{CK(\overline{a},\overline{b},M\restriction_D)}$.
- The indices for wellorderings relative to an oracle X are not a $\Sigma_1^1(X)$ set. So $\theta(e)$ holds for some e with $\Phi_e^{\overline{a},\overline{b},M\restriction_D}$ ill-founded. So there is an autoisometry of M taking $\overline{a}$ to $\overline{b}$.

Proposition (Strengthens result for rigid of Chan 2016)

Suppose that the isometry relation on tuples of the same length in M is Δ^1_1. Then $\mathsf{SR}(M)$ is computable in $M\restriction_D$.

Proof. The following property of $e\in\omega$ is Σ^1_1:

$\Phi_e^{M\restriction_D}$ codes a linear order L such that $\exists n\exists\bar{a},\bar{b}\in M^n$

$$[\bar{a}\not\approx\bar{b}\ \wedge\ \text{Player 2 has a winning strategy in } G_L^M(\bar{a},\bar{b},D).]$$

For each such e, $\Phi_e^{M\restriction_D}$ is a well-ordering.
By Σ^1_1 bounding, the set of such $\Phi_e^{M\restriction_D}$ is then bounded by an ordinal computable in $M\restriction_D$.

This hypothesis on the complexity of the isometry relation is not necessarily satisfied. Melleray (2015) has constructed a computable metric space where the isometry relation of elements is not Borel; it is in fact Borel-complete for OER of continuous Polish group actions.

People's views on the question: is the Scott rank of every Polish metric space countable?

- Yes
- No
- Independent of ZFC
- Wrong question!
  infinitary classical logic is too expressive, and hence the wrong language for Polish metric spaces anyway.

For example, local compactness of Polish metric spaces can be expressed in infinitary classical logic, but is not a Borel property (N. and Solecki, CiE 2015).

Complexity of low-level Scott relations

- For a computable structure M the Scott relations r_n^M are in Π^0_{2n}.
- In a computable metric space M, the players play reals, so the Scott relations r_n^M are merely in Π^1_{2n}.
- The following shows that in the metric setting, even the level-1 Scott relation aren't Borel in general.

Proposition (with P. Schlicht, Logic Blog 2017)

There is a computable Polish metric space M and a computable sequence $\langle y_n \rangle_{n \in \mathbb{N}}$ of elements of M such that the set

$$\{(m,k) \in \omega \times \omega \mid r_1^M(y_m, y_k)\}$$

is Π^1_2-complete.

So, is there a continuous version of the Scott relations?

Continuous metric Scott analysis

Joint work with Ben Yaacov, Doucha, Tsankov (2017)

The α, ϵ–game

- We are given for each $k \geq 1$ similarity functions r_0 defined on tuples $\overline{a} \in A^k, \overline{b} \in B^k$ with values in $\mathbb{R}_{\geq 0}$.
- They measure how $\overline{a}$ behaves similarly to $\overline{b}$ within the context of the Polish metric structures. They are all Lipschitz.
- E.g. $r_0(\overline{a}, \overline{b}) = \max_{i,k<n} |d_A(a_i, a_k) - d_B(b_i, b_k)|/2$.

- For bounded Polish metric spaces A, B, an ordinal α and $\overline{a} \in A^n, \overline{b} \in B^n$, $\varepsilon > 0$, the game $G^{A,B}_{\alpha,\varepsilon}(\overline{a}, \overline{b})$ is played almost as before (for $D = M$), except that the winning condition is different:
- After round i, if $r_0(a_0 a_1 \dots a_{n+i}, b_0 b_1 \dots b_{n+i}) > \epsilon$, then Player 1 wins.

The Scott function $r_\alpha(\overline{a}, \overline{b})$ is the infimum of the ϵ so that Player 2 has a winning strategy in $G^M_{\alpha,\varepsilon}(\overline{a}, \overline{b})$.

Explicit definition of Scott functions

One checks by induction on ordinals that

$$\begin{aligned} r_{\alpha+1,n}(\bar{a},\bar{b}) &= \max\big(\sup_{x\in A}\inf_{y\in B} r_{\alpha,n+1}(\bar{a}x,\bar{b}y), \sup_{y\in B}\inf_{x\in A} r_{\alpha,n+1}(\bar{a}x,\bar{b}y)\big) \\ r_{\alpha,n}(\bar{a},\bar{b}) &= \sup_{\beta<\alpha} r_{\beta,n}(\bar{a},\bar{b}), \quad \text{for } \alpha \text{ limit or } \alpha=\infty. \end{aligned}$$

- The r_α are uniformly continuous, and nondecreasing in α.
- So the least ordinal $\alpha_{A,B}$ with $r_\alpha(.,.) = r_{\alpha+1}(.,.)$, for each length and each pair of tuples of the same length, is countable.
- The ordinal $\alpha_{A,B}$ is the continuous r_0-Scott rank of the pair A, B.

Similarity function for isometry

Given bounded Polish metric spaces A, B.
For $\overline{a} \in A^n, \overline{b} \in B^n$, let

$$r_0(\overline{a}, \overline{b}) = \max_{i<k<n} i \cdot |d_A(a_i, a_k) - d_B(b_i, b_k)|.$$

Theorem

- With this choice of the ground Scott functions r_0, $r^{A,B}_{\alpha_{A,B},n}(\bar{a}, \bar{b})$ is the infimum of the distances $\sum_{i<n} i \cdot d(f(\overline{a}_i), \overline{b}_i)$ where $f\colon A \cong B$.
- In particular, letting $n = 0$, the value $r^{A,B}_{\alpha_{A,B},0} = 0$ means that the two spaces are isometric.

Similarity function for Gromov-Hausdorff distance

Given bounded Polish metric spaces A, B.

- Gromov (1999) defined $d_{GH}(A, B)$ as the infimum of the Hausdorff distances of isometric embeddings of A, B into a third metric space.
- In general this can be 0 without the spaces being isometric.
- For compact spaces, and for spaces with a lower bound on the distance of two different points, having GH-distance 0 implies isometric.
- So "$d_{GH}(A, B) = 0$" is not Borel (since isometry of such discrete spaces is complete for S_∞ orbit relations).

For $\overline{a}, \overline{b} \in M^n$, let $r_0(\overline{a}, \overline{b}) = \max_{i,k<n} |d(a_i, a_k) - d(b_i, b_k)|/2$.

Theorem

With this choice of the ground Scott functions r_0, $r^{A,B}_{\alpha_{A,B},0}$ is the Gromov-Hausdorff distance of A, B.

A bit of infinitary continuous logic

still joint work with Ben Yaacov, Doucha, Tsankov (2017)

Structures, moduli

- Metric structures M
- metric predicates of arity k are bounded uniformly continuous $\mathbb{R}$-valued functions on M^k
- distance is a metric predicate of arity 2, replacing $=$

A k-ary modulus of continuity is a certain nice function $\Delta\colon [0,\infty)^k \to [0,\infty)$.

A function $f\colon M^k \to \mathbb{R}$ respects Δ if for each $\overline{x}, \overline{y} \in M^k$

$$|f(\overline{x}) - f(\overline{y})| \leq \Delta(d(x_i, y_i)_{i<k}).$$

Each predicate symbol R goes with a modulus Δ_R of the same arity.

Formulas

- Start from basic formulas such as $\hat{d}(x, y)$ interpreted as $\min(1, d(x, y))$.
  (The possible values have to be bounded.)
- The quantifiers over a variable x_i are $\sup_{x_i} \phi$ and $\inf_{x_i} \phi$.
- Semantics:
  for each metric L-structure M, formula $\phi(\overline{x})$ and $\overline{a} \in M^k$, we have a real value $\phi^M(\overline{a})$.
- Example with $k = 0$: let $\phi = \sup_x \sup_y \hat{d}(x, y)$.
  What is ϕ^M?
  Answer: The minimum of 1 and the diameter of M.

Theorem (Ben Yaacov, Nies, Tsankov; http://arxiv.org/abs/1407.7102)

Let $U\colon \mathcal{M}_L \to \mathbb{R}$ be a bounded Borel function that is isometry-invariant. There exists a continuous $\mathcal{L}_{\omega_1,\omega}(L)$-sentence ϕ such that

$$U(A) = \phi^A,$$

for all $A \in \mathcal{M}$.

Example 1: $U(M) = \min(1, \mathrm{diam}(M))$.

Example 2: Fix Polish L-structure M.

- Let $U(A) = 0$ if $A \cong M$, and $U(A) = 1$ otherwise.
- The function U is Borel by Elliott, Farah, Paulsen, Rosendal, Toms, Tornquist (2013).
- Then ϕ is a continuous Scott sentence for M.

Weak moduli

Let N be a natural number or $\mathbb{N}$. A weak modulus of arity N is a function $\Omega\colon [0,\infty)^n \to [0,\infty]$ that is:

1. non-decreasing, subadditive, vanishing at zero:

$$\Omega(\delta) \leq \Omega(\delta + \delta') \leq \Omega(\delta) + \Omega(\delta'), \qquad \Omega(0) = 0;$$

2. lower semi-continuous in the product topology and separately continuous in each argument.

Examples. The unbounded weak modulus $\Omega^U\colon [0,\infty)^{\mathbb{N}} \to [0,\infty]$ is defined by

$$\Omega^U(\delta) = \sup_i \, i \cdot \delta_i.$$

The 1-Lipschitz weak modulus $\Omega^L\colon [0,\infty)^{\mathbb{N}} \to [0,\infty]$ is defined by

$$\Omega^L(\delta) = \sup_i \delta_i, \quad \text{where } \delta = (\delta_0, \delta_1, \ldots).$$

Ω-formulas

- All basic formulas $\phi(x_0, \ldots, x_{n-1})$ that only depend on the first n variables and respect Ω are n-ary Ω-formulas.
- If $\{\phi_i : i \in \mathbb{N}\}$ are n-ary Ω-formulas, then $\bigvee_i \phi_i$ and $\bigwedge_i \phi_i$ are n-ary Ω-formulas.
- If ϕ is an $(n+1)$-ary Ω-formula, then $\inf_{x_n} \phi$ and $\sup_{x_n} \phi$ are n-ary Ω-formulas.
- If $\phi_0, \ldots, \phi_{k-1}$ are n-ary Ω-formulas and $f \colon \mathbb{R}^k \to \mathbb{R}$ is a 1-Lipschitz function, then $f(\phi_0, \ldots, \phi_{k-1})$ is a n-ary Ω-formula.
- An Ω-sentence is a 0-ary Ω-formula.

Fix a signature L and a weak modulus $\Omega\colon [0,\infty)^{\mathbb{N}} \to [0,\infty]$.

Let α be an ordinal or the symbol ∞ greater than all ordinals. Let $n \in \mathbb{N}$, let A and B be structures and let $\bar{a} \in A^n$, $\bar{b} \in B^n$. We define the back-and-forth pseudo-distance $r_{\alpha,n}^{A,B,\Omega}(\bar{a},\bar{b})$ by induction on α:

$$r_0^{A,B}(\bar{a},\bar{b}) = \sup_{\phi} \left|\phi^A(\bar{a}) - \phi^B(\bar{b})\right|,$$

where ϕ varies over all basic n-ary Ω-formulas with $I_\phi \subseteq [0,1]$. For α limit (or ∞),

$$r_\alpha^{A,B}(\bar{a},\bar{b}) = \sup_{\beta<\alpha} r_\beta^{A,B}(\bar{a},\bar{b}).$$

For the successor step,

$$r_{\alpha+1}^{A,B}(\bar{a},\bar{b}) = \sup_{c\in A,\, d\in B} \inf_{c'\in A,\, d'\in B} r_\alpha^{A,B}(\bar{a}c,\bar{b}d') \vee r_\alpha^{A,B}(\bar{a}c',\bar{b}d).$$

Lemma

1. *For fixed* α *and* n, r_α *is a pseudo-distance on the class of all pairs* $A\bar{a}$ *bounded by* 1.
2. *For fixed* α, n, A, *and* B, *the function* $r_\alpha^{A,B}$ *is uniformly continuous on* $A^n \times B^n$, *respecting the modulus* $\Omega|_n$ *on each side.*

Lemma

1. *If* $\beta < \alpha$ *then* $r_\beta \leq r_\alpha$ *(i.e.,* $r_{\beta,n} \leq r_{\alpha,n}$ *for all* n*);*
2. *If* κ *is an infinite cardinal and* A *and* B *are structures of density character at most* κ, *then there exists* $\alpha < \kappa^+$ *such that* $r_{\alpha+1}^{A,B} = r_\alpha^{A,B}$. *Moreover, in this case, the sequence of* $r^{A,B}$ *stabilizes beyond* α, *i.e.,* $r_\infty^{A,B} = r_\alpha^{A,B}$.

Define quantifier rank in the expected way.

Proposition

Let α be an ordinal, $A, B \in \mathcal{M}$, $\bar{a} \in A^n$ and $\bar{b} \in B^n$. Then

$$r_\alpha^{A,B}(\bar{a}, \bar{b}) = \sup_\phi \left|\phi^A(\bar{a}) - \phi^B(\bar{b})\right|, \tag{1}$$

where ϕ varies over all n-ary Ω-formulas of quantifier rank at most α.

Ω-Scott sentences (1)

Let A be separable. For $\alpha = 0$,

$$\phi_{0,n,A\bar{a}} = \bigvee_{\phi} \left|\phi^A(\bar{a}) - \phi(x_0, \ldots, x_{n-1})\right|,$$

as ϕ varies over a countable dense family of basic n-ary Ω-formulas. For α limit,

$$\phi_{\alpha,n,A\bar{a}} = \bigvee_{\beta<\alpha} \phi_{\beta,n,A\bar{a}}.$$

For a successor,

$$\phi_{\alpha+1,n,A\bar{a}} = \left(\bigvee_{c\in\mathbb{N}} \inf_{x_n} \phi_{\alpha,n+1,A\bar{a}c}\right) \vee \left(\sup_{x_n} \bigwedge_{c\in\mathbb{N}} \phi_{\alpha,n+1,A\bar{a}c}\right).$$

Ω-Scott sentences (2)

Now let α_A be the Ω-Scott rank of A. Since A is separable, $\alpha_A < \omega_1$. We define σ_A, the Scott sentence of A, as

$$\sigma_A = \phi_{\alpha_A,0,A} \vee \bigvee_{n,\,\bar{a}\in\mathbb{N}^n} \sup_{x_0,\ldots,x_{n-1}} 1/2\left|\phi_{\alpha_A,n,A\bar{a}} - \phi_{\alpha_A+1,n,A\bar{a}}\right|.$$

This is an Ω-sentence; the coefficient $1/2$ is needed because the function $(x_1, x_2) \mapsto |x_1 - x_2|$ is 2-Lipschitz and in Ω-formulas, we only allow 1-Lipschitz connectives.

Theorem

Let B be a separable structure. Then $B \models (\sigma_A = 0)$ iff $r_\infty(A, B) = 0$.

First application

Recall: The unbounded weak modulus $\Omega^U : [0,\infty)^{\mathbb{N}} \to [0,\infty]$ is defined by

$$\Omega^U(\delta) = \sup_i \, i \cdot \delta_i.$$

- We show that $r_\infty^{\Omega^U}(A,B)$ denotes isomorphism of metric structures: it is 0 if $A \cong B$, and 1 otherwise.
- So we get a continuous Scott sentence for A.

Second application

Recall: The 1-Lipschitz weak modulus $\Omega^L \colon [0,\infty)^{\mathbb{N}} \to [0,\infty]$ is defined by

$$\Omega^L(\delta) = \sup_i \delta_i, \quad \text{where } \delta = (\delta_0, \delta_1, \ldots).$$

- We show that $r_\infty^{\Omega_L}(A, B)$ is the Gromov-Hausdorff distance d_{GH} of A, B.
- For pure metric spaces, this is the inf of Hausdorff distances of isometric embeddings of A, B into a third metric space C.

Corollary

For each separable A, the set $\{B \colon d_{GH}(A, B) = 0\}$ is Borel.

References

- Friedman, Fokina, Koerwien, N., The Scott rank of Polish metric spaces (2012), arxiv.org/abs/1906.00351.
- Ben Yaacov, Doucha, Nies, Tsankov. Metric Scott Analysis. Advances in Mathematics 318 (2017) 46 – 87.
- William Chan manuscripts (2016)
- Logic Blog 2017, available from my main web page.
- These slides on the IMS web page.
- A. Montalbán. A robuster Scott rank, Proc. Amer. Math. Soc., 143 (2015), 5427–5436.