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For countable structures $\mathcal{A}, \mathcal{B}$, say:

- **(Muchnik/weak reducibility)** $\mathcal{A} \leq_w \mathcal{B}$ *iff for every* ω*-copy* B *of* $\mathcal{B}$ *there is an* ω*-copy* A *of* $\mathcal{A}$ *with* $B \geq_T A$*.*
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If $\omega_1^L = \omega_1$ then there is an embedding of ω_1 into the countable ordinals under uniform *hyperarithmetic* reducibility (via Hamkins-Linetsky-Reitz).

Small (cheap) antichains

Definition (Generic strong reducibility)

For structures $\mathcal{A}, \mathcal{B}$ of arbitrary cardinality, write $\mathcal{A} \leq_s^* \mathcal{B}$ iff $\mathcal{A} \leq_s \mathcal{B}$ in every forcing extension where each is countable.

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Now note that $f(M) > \theta^M = \theta^N = \theta$ for every $M, N \in \mathbb{K}$.

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A *local reducibility notion* is a relation $R \subseteq \omega \times Struc \times Struc$ (where *Struc* is the set of structures with domain ω topologized as usual) such that

- If $R(a, X, Y)$ and $R(b, Y, Z)$ then $R(c, X, Z)$ for some c.
- For each X there is some e with $R(e, X, X)$.
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Corollary (ZFC+LC)

For $\trianglelefteq$ any projective uniform reducibility, $\exists$ club $\trianglelefteq$-antichain of countable ordinals.

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Will use stronger hypotheses later.

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G $\mathbb{P}$-generic over V: a structure $M \in V[G]$ is generically presentable *if $M = \nu[G]$ for some ν with*

$$\Vdash_{\mathbb{P}^2} \nu[H_0] \cong \nu[H_1].$$

Especially interested in "countable" g.p.s.s: $\Vdash_{\mathbb{P}} |\nu[G]| = \aleph_0$.

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Can talk about $\trianglelefteq^*$, indices, etc. for generically presentable structures; this is enough:

Theorem (Knight-Montalbán-S., folklore?)

If $T \in V$ a VC-counterexample and $M \in V[G]$ a model of T, then M is generically presentable over V.

VC-counterexamples have short degree structures

Proposition (ZFC+LC)

If T is a VC-counterexample and $\trianglelefteq$ is a projective uniform reducibility on the countable models of T, then

$$\{otp(\{\alpha \in \omega_1 : \alpha \leq_s \mathcal{A}\}) : \mathcal{A} \models T \text{ countable}\}$$

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Corollary (ZFC+LC)

If T is a counterexample to Vaught's conjecture and $\trianglelefteq$ is a projective uniform reducibility, then there is a bound below ω_1 on the ordinals which embed into the $\trianglelefteq$-degrees of countable models of T.

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Transform a counterexample $\trianglelefteq$ to a counterexample to the above proposition via a pre-well-ordering of the countable models of T of length ω_1.

Digression 1: well-ordering VC-counterexamples

Missing from the above is an analogue of the “constant-on-a-club” fact.

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Is there a sense in which, for T a VC-counterexample, $IO_{\trianglelefteq}(-)$ is constant on "most" of $Mod(T)$? Easiest approach to "yes" would be to get a well-ordering of $Mod(T)$.

Question

If T is a VC-counterexample, is there a projective well-ordering of the countable models of T (as opposed to a countable-to-one pre-well-ordering)?

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Generically presentable *structures up to isomorphism* are nontrivial:

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Question

Is there a generically presentable cardinality $(\nu, \mathbb{P})$ such that $\forall X \in V, \Vdash_{\mathbb{P}} \nu[G] \not\equiv X$? In ZFC trivially no; over ZF? In canonical models where choice fails? What if we require $\nu[G] \subseteq \mathbb{R}^{V[G]}$?

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The uniform degrees of models of T coming from $\trianglelefteq$ embed ω_1 and the reducibility is Σ_1^1.

Corollary (ZFC+LC)

If T is a countable first-order theory with uncountably many countable models, then T is a VC-counterexample iff every Σ_1^1 uniform reducibility notion yields a degree structure on models of T into which ω_1 does not embed.

A similar construction gives a projective uniform reducibility $\trianglelefteq$ on structures such that

$$\{\mathit{ordertype}(\{\alpha \in \omega_1 : \alpha \trianglelefteq \mathcal{A}\}) : \mathcal{A} \models T\}$$

is cofinal in ω_1, so that yields another equivalence.

Digression 3: "near" miss counterexamples

If T has a perfect set of models, the uniform reducibility above does not seem to come from a Δ^1_1 relation on individual copies: why should the perfect set P of nonisomorphic models contain a representative of *every* model?

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For Γ *a pointclass, a theory* T *with uncountably many countable models is* Γ-short *iff for every uniform reducibility notion* $\trianglelefteq$ *coming from a local reducibility notion in* Γ*, there is no embedding of* ω_1 *into the* $\trianglelefteq$*-degrees of models of* T*.*

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Is there a Δ^1_1-short theory?

It is even unclear to me whether Π^1_1-short theories (other than VC-counterexamples) can exist — I suspect not.

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Unlike weak reducibility, it's easy to show that there are many "phase transitions" for strong reducibility on ordinals: for a club of countable ordinals α, have

- $\{\beta < \alpha : \beta \leq_s \alpha\}$ bounded below α.
- $\min\{\beta > \alpha : \beta \not\leq_s \alpha\} < \alpha \cdot 2$.

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Question

What more can we say about $S(\alpha)$?

Forcing over admissibles

An *admissible set* is a transitive A satisfying KP ($+$ Inf): basic set theory axioms $+$ Σ_1-replacement $+$ Δ_1-separation.

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Theorem (Barwise; see also Ershov, Jensen, Zachar)

If A is an admissible set and $\mathbb{P} \in A$ is a forcing notion, then for G $\mathbb{P}$-generic over A the extension $A[G]$ is also admissible.

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$$\sigma := otp(\sum_{e \in X} \Phi_e^A)$$

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Can now check that everything relevant was absolute to L, so this goes through in ZFC alone. First sentence of theorem now follows from same proof for α "sufficiently ω_1-like" (admissible + "every subclass of ω is a set" is enough).

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Proposition (Platek)

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Proof uses an extension of Σ^1_1-bounding to structures,

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Proof uses an extension of Σ_1^1-bounding to structures, and relies on the fact that we can effectively build a tree whose branches code definable expansions of L_α from an ω-copy of α.

Thanks!