The complexity of hyperarithmetic isomorphism

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The motivating question:

How difficult is it to tell if two structures are isomorphic?

Index set approach

Let $\mathcal{M}_0, \mathcal{M}_1, \ldots$ be an enumeration of the computable structures. The set

$$\{(e, i) : \mathcal{M}_e \cong \mathcal{M}_i\}$$

is Σ^1_1-complete.

Reductions of equivalence relations

Let E be an equivalence relation on X, and F be an equivalence relation on Y. A *reduction* of E to F is a nice function $f\colon X \to Y$ which induces an embedding of the equivalence classes

$$X/E \hookrightarrow Y/F.$$

That is, for all $x, z \in X$,

$$x \, E \, z \quad \Leftrightarrow \quad f(x) \, F \, f(z).$$

In descriptive set theory

- X and Y are Polish spaces and “nice” means Borel.

Theorem (Friedman-Stanley,1989)

Isomorphism of countable structures is not complete among $\mathbf{\Sigma}^1_1$ equivalence relations.

In computability

- $X = Y = \mathbb{N}$ and “nice” means computable.

Computable reducibility of equivalence relations investigated by, among others,

- Ershov (1977)
- Bernardi and Sorbi (1983)
- Gao and Gerdes (2001)
- Coskey, Hamkins, R. Miller (2012)
- Andrews, Lempp, J. Miller, Ng, San Mauro and Sorbi (2014)
- Ianovski, R. Miller, Ng, Nies (2014)

The complexity of isomorphism

Theorem (Fokina,S.Friedman,Harizanov,Knight,McCoy,Montalbán)
Isomorphism of computable structures is complete for Σ^1_1 equivalence relations on $\mathbb{N}$ (under computable reduction).

What about definable isomorphisms?

We can refine the question and ask: how difficult is it to tell if there is a *definable* isomorphism between two computable structures?

For example, we can ask whether there is a computable isomorphism between two given computable structures.

Theorem (Fokina,S.Friedman,Nies)

Computable isomorphism on computable structures is complete for Σ^0_3 *equivalence relations on* $\mathbb{N}$.

Hyperarithmetic isomorphism

Recall (Spector-Gandy) that the quantifier

$$(\exists f \in \mathrm{HYP})\,\varphi$$

(where φ is arithmetic) gives a Π^1_1 relation.

Theorem

Hyperarithmetic isomorphism on computable structures is complete for Π^1_1 equivalence relations on $\mathbb{N}$.

A sketch

Enumerating Π^1_1 equivalence relations

Π^1_1 subsets of $\mathbb{N}$ are $\Sigma_1(L_{\omega_1^{\mathrm{ck}}})$-definable, which makes them analogous to c.e. sets: they get enumerated in a process that takes ω_1^{ck} many steps.

If E is a Π^1_1 equivalence relation then E is the union $\bigcup_{\alpha<\omega_1^{\mathrm{ck}}} E_\alpha$ of increasing equivalence relations, with each E_α hyperarithmetic, uniformly in α.

Π^1_1 Completeness: the plan

Given a Π^1_1 equivalence relation E, we build structures $\mathcal{M}_n$ for $n \in \mathbb{N}$.

- If $m \, E \, n$ then eventually, all decisions we make for $\mathcal{M}_m$ are identical to those we make for $\mathcal{M}_n$.
- If $m \not\!E \, n$ then we actively diagonalise against all possible hyperarithmetic isomorphisms.

Components and isomorphisms

Each structure $\mathcal{M}_n$ consists of disjoint components indexed by pairs (e,k). The component (e,k) is used to (possibly) diagonalise against the e^{th} hyperarithmetic isomorphism from $\mathcal{M}_k$ to $\mathcal{M}_n$.

Each component (e,k) consists of two linear orderings, $(A_{e,k})^{\mathcal{M}_n}$ and $(B_{e,k})^{\mathcal{M}_n}$. These are not a-priori distinguished from each other, but the structures are designed so that any isomorphism $f\colon \mathcal{M}_k \to \mathcal{M}_n$ must either

- restrict to isomorphisms

$$(A_{e,k})^{\mathcal{M}_k} \cong (A_{e,k})^{\mathcal{M}_n} \quad \& \quad (B_{e,k})^{\mathcal{M}_k} \cong (B_{e,k})^{\mathcal{M}_n};$$

  or
- restrict to isomorphisms

$$(A_{e,k})^{\mathcal{M}_k} \cong (B_{e,k})^{\mathcal{M}_n} \quad \& \quad (B_{e,k})^{\mathcal{M}_k} \cong (A_{e,k})^{\mathcal{M}_n}.$$

Π^1_1 events

There are two pieces of relevant information that may be given to us:

- If the e^{th} partial Π^1_1 function $\varphi_e \colon \mathbb{N} \to \mathbb{N}$ is an isomorphism from $\mathcal{M}_k$ to $\mathcal{M}_n$, then at some level $\alpha < \omega_1^{\text{ck}}$, we will be told if φ_e matches the A's and the B's, or flips between them.
- If $k\,E\,n$ then at some level $\alpha < \omega_1^{\text{ck}}$ we will discover this fact.

The question: if both happen, which happens first?

Diagonalising and copying

Suppose that at some level $\alpha < \omega_1^{\mathrm{ck}}$ we discover that φ_e matches A's with A's and B's with B's.

- if $k \not E_\alpha n$ then we want to ensure that $(A_{e,k})^{\mathcal{M}_k} \not\cong (A_{e,k})^{\mathcal{M}_n}$ and the same for the B's.
- If $k E_\alpha n$ then we want to make $(A_{e,k})^{\mathcal{M}_k} \cong (A_{e,k})^{\mathcal{M}_n}$ and the same for the B's.

We will let $(A_{e,k})^{\mathcal{M}_n} \cong \omega^\alpha$ and $(B_{e,k})^{\mathcal{M}_n} \cong \omega^\alpha \cdot 2$ if $k E_\alpha n$, and flip otherwise.

Why would this work?

Suppose that $n\,E\,m$. We want to show that $\mathcal{M}_n \cong \mathcal{M}_m$ via a hyperarithmetic isomorphism.

Suppose that $n\,E_\beta\,m$ for some $\beta < \omega_1^{\mathrm{ck}}$. We construct an isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_m$ using (roughly) $\varnothing^{(\beta)}$. Fix a pair (e, k).

- If φ_e has declared its intentions at level $\alpha < \beta$, then $\varnothing^{(\beta)}$ knows this fact and can construct the isomorphism between the orderings which have ordertype ω^α and those which have ordertype $\omega^\alpha \cdot 2$.
- If not, then whatever happens later, we know that $(A_{e,k})^{\mathcal{M}_n} \cong (A_{e,k})^{\mathcal{M}_m}$ and the same for the B's, and we will in fact ensure that $\varnothing^{(\beta)}$ can compute the isomorphisms.

Questions

- Why ω^α and $\omega^\alpha \cdot 2$? Why not 1 and 2?
- What happens if φ_e is partial?
- Why are the structures *computable*?

True stages

An answer

We use an iterated priority argument to approximate the entire construction.

- Level α uses the information provided by $\varnothing^{(\alpha)}$ to construct the α^{th} Hausdorff derivatives of the $A_{e,k}$'s and $B_{e,k}$'s.
- If we diagonalise at level α then these derivatives are either 1 or 2.
- If $n\,E_\alpha m$ then $\varnothing^{(\alpha)}$ builds an isomorphism between $(A_{e,k})_\alpha^{\mathcal{M}_n}$ and $(A_{e,k})_\alpha^{\mathcal{M}_m}$.
- At stage $s < \omega$ we use our current (finite) guess about each $\varnothing^{(\alpha)}$ to build accordingly.

The challenge:

- If at stage s we were wrong about $\varnothing^{(\alpha)}$, we need to fix our mistakes. On the other hand, at the same stage, we may have been right about $\varnothing^{(\beta)}$ for some $\beta < \alpha$, and we need to preserve what we built at that level.

Iterated priority arguments

Developed by both Harrington and Ash-Knight.

Montalbán gave a dynamic version of the Ash-Knight machinery which allows for level-by-level control. He defines:

- For each α and $s < \omega$, a finite approximation ∇^α_s of $\varnothing^{(\alpha)}$ at stage s.
- The fundamental notion $s \leqslant_\alpha t$: s appears α-*true* at stage t – roughly, if ∇^α_t extends ∇^α_s.
- s is α-*true* if $s \leqslant_\alpha \omega$, where ∇^α_ω is essentially $\varnothing^{(\alpha)}$.

True stages, directly

We have discovered another approach to developing the machinery of α-true stages. The main idea: rather than first define ∇^α_s and based on these, the relations $s \leqslant_\alpha t$, to define both by simultaneous recursion on α.

- $\leqslant_0$ is the usual ordering on $\mathbb{N}$.
- Given $\leqslant_\alpha$, ∇^α_t is the increasing enumeration of the stages $s <_\alpha t$.
- Given ∇^α_t, we can define $s \leqslant_{\alpha+1} t$ using non-deficiency stages: $s \leqslant_{\alpha+1} t$ if $s \leqslant_\alpha t$ (so $\nabla^\alpha_s \preccurlyeq \nabla^\alpha_t$), and if y is the number enumerated into $(\nabla^\alpha_s)'$ at stage s, then no number smaller than y has since been enumerated into $(\nabla^\alpha_t)'$.
- For limit λ, $s \leqslant_\lambda t$ iff $(\forall \alpha < \lambda)\ s \leqslant_\alpha t$.

Existence of true stages

The main question is: why should there be any α-true stages for $\alpha \geqslant \omega$?

We concentrate on the case $\alpha = \omega$.

We actually need to modify the definition:

- ∇_t^n is the increasing enumeration of $\{s > n : s <_n t\}$.

Then:

- The first n-true stage after stage n is ω-true.

The point is that the empty oracle cannot be wrong about its jump: it makes no commitments.

There is a reason why this resembles diagonal intersections.

A problem

What is α? We need to work with a computable copy. Thus, the machinery only works when we fix a computable ordinal $\delta < \omega_1^{\mathsf{ck}}$ and define $s \leqslant_\alpha t$ for all $\alpha \leqslant \delta$. This doesn't take us all the way up to ω_1^{ck}.

Overspill

Overspill

Perform the entire construction inside an ω-model of ZFC which omits ω_1^{ck}.

In that universe V^*, we fix a computable pseudo-ordinal δ^*, and develop the true stage machinery. We also:

- approximate the Π^1_1 equivalence relation E^* inside V^*; and
- diagonalise against partial Π^1_1 functions in the sense of V^*.

V^* is arithmetically, and hence hyperarithmetically, absolute. This means:

- $E^*_\alpha = E_\alpha$ for $\alpha < \omega_1^{\mathrm{ck}}$;
- if φ_e is total then it equals φ^*_e (and we see convergence at a well-founded level).

We note that if φ^*_e reveals itself at an ill-founded level, then the linear orderings $A_{e,k}$ and $B_{e,k}$ are Harrison.

Extending beyond ω_1^{ck} does not trouble us

- If $n\,E\,m$ then $n\,E^*_\alpha\,m$ for some $\alpha < \omega_1^{ck}$. Thus, $(\varnothing^{(\alpha)})^*$ computes an isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_m$. But since α is standard, $(\varnothing^{(\alpha)})^* = \varnothing^{(\alpha)}$.

- If $n \not{E}\, m$ then for all $\alpha < \omega_1^{ck}$, $n \not{E}^*_\alpha\, m$. It is possible that $n\,E_*\,m$. Then $(\varnothing^{(\beta)})^*$ computes an isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_m$ for some ill-founded β; but $(\varnothing^{(\beta)})^*$ is not hyperarithmetic (indeed it computes all hyperarithmetic sets).

  Since we do not see that $n\,E^*_\alpha\,m$ at any well-founded α, the construction successfully diagonalises against all φ^*_e which declare themselves at a well-founded stage; this includes all hyperarithmetic maps.

Thank you.