Randomness for infinite time Turing machines

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Overview

Π^1_1-randomness

Definition

- Δ^1_1-random: avoids all Δ^1_1 null sets
- Π^1_1-ML-random: avoids maximal Π^1_1-coded $\mathbf{\Pi}^0_2$ set $\bigcap_n U_n$ with $\lambda(U_n) \leq 2^{-n}$
- Π^1_1-random: avoids maximal Π^1_1 null set

Membership of a real x in a Π^1_1 set A is given by a recursion with $\omega_1^{ck,x}$ stages.

Π^1_1-randomness

Fact (Stern; Chong, Nies, Yu)

A real x is Π^1_1-random if and only if x is Δ^1_1-random and $\omega_1^x = \omega_1^{ck}$.

Proof sketch.

$\Longrightarrow$: Let $A = \{x \mid \omega_1^x > \omega_1^{ck}\}$.

A is Π^1_1: $x \in A$ is witnessed in $\omega_1^{ck} < \omega_1^x$ stages.

A is null: Sufficiently random reals x preserve "ω_1^{ck} is x-admissible".

$\Longleftarrow$: Let A be a Π^1_1 null set.

Since $\omega_1^{ck} = \omega_1^x$, $x \in A$ is witnessed in $\alpha < \omega_1^{ck}$ stages. Restricting to α stages yields a Borel set $B \subseteq A$ with code in $L_{\omega_1^{ck}}$ and $x \in B$. □

Π^1_1-randomness

If p is a Borel code, let $[p]$ denote the decoded set.

Random forcing consists of all Borel codes p with $\lambda([p]) > 0$.

Fact

Let α be admissible. A real x is set-generic over L_α for random forcing if and only if x avoids every Borel null set with code in L_α.

Definition

Let α be a limit ordinal. A real x is α-random (=random over L_α) if it avoids every Borel null set with a code in L_α.

Thus ω_1^{ck}-random equals Δ^1_1-random.

Π^1_1-randomness

Theorem (Yu)

If x is class-generic over $L_{\omega_1^{ck}}$ for random forcing, then x is Δ^1_1-random, but not Π^1_1-random. Thus $\omega_1^x > \omega_1^{ck}$.

In fact, x is not Π^1_1-ML-random.

Proof sketch.

Let $\vec{U} = \langle U_n \mid n \in \omega \rangle$ be a universal Π^1_1-ML-test and $A = \bigcap_n U_n$.

For each n, $D_n = \{p \mid [p] \subseteq U_n\}$ is dense in random forcing: since every Δ^1_1 set of positive measure contains a Δ^1_1 real.

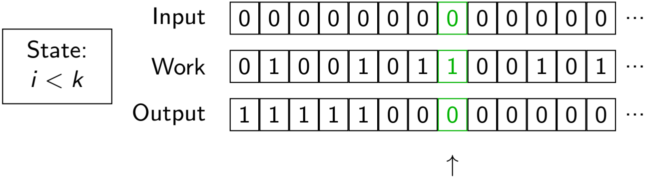
Since the D_n's are definable over $L_{\omega_1^{ck}}$, any class-generic filter G over $L_{\omega_1^{ck}}$ will have $G \cap D_n \neq \emptyset$ for all n.

Let x be the generic real given by G. Then $x \in U_n$ for all n. Thus $x \in A$. $\square$

ITTMs

An infinite time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers:

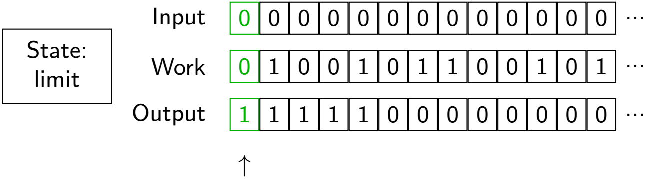
- The input tape
- The output tape
- The working tape



ITTMs

It behaves like a standard Turing machine at successor steps of a computation.
At limit steps of computation:

- The head goes back to the first cell.
- The machine goes into a "limit" state.
- The value of each cell equals the lim inf of the values at previous stages of computation.



Definition

A set of reals A is decidable if there is an ITTM M such that $M(x)\downarrow = 1$ if $x \in A$ and 0 if $x \notin A$.

Proposition (Hamkins, Lewis)

The sets of reals coding a well-order $\prec$ (with the code $x = \{(m, n) \mid m \prec n\}$) is decidable.

The strength of ITTMs

Proposition (Hamkins, Lewis)

The set of reals coding a well-order is decidable.

Proof sketch.

The algorithm is as follows.

- *Search for an infinite decreasing sequence by choosing some* x_0, $x_1 < x_0$ *etc. If this is successful, halt with state* 0. *Otherwise the search terminates with the least element. Remove this and start over.*
- *If all elements are removed at some point, halt with state* 1.

Suppose it's not a well-order. Then the well-founded part is removed at some time. After this, the search for an ill-founded sequence takes ω steps. □

Corollary (Hamkins, Lewis)

Every Π^1_1 *set is decidable.*

The strength of ITTMs

Definition

A real x is writable if some ITTM runs on empty input and halts with output x.

Definition

An ordinal is writable if it has a writable code.

Definition (Hamkins, Lewis)

Let λ denote the supremum of all writable ordinals.

The strength of ITTMs

Proposition (Hamkins, Lewis)

There is an ITTM which writes λ on its output tape, then leaves the output tape unchanged without ever halting.

Proof sketch.

Run a "universal" machine U that simulates all other machines.

In each step i, calculate the sum α_i of all ordinals that have been written by halting programs, in the order of their appearance.

If i is the supremum of halting times, $\alpha_i = \alpha_j$ for all $i \leq j$. Then $\alpha_i \geq \lambda$. □

The strength of ITTMs

Definition (Hamkins, Lewis)

A real is eventually writable if there in an ITTM and a step i such that the real is on the output tape does not change after step i.

Let ζ be the supremum of the eventually writable ordinals.

Definition (Hamkins, Lewis)

A real is accidentally writable if there in an ITTM and a step i such that the real is on the output tape at time i.

Let Σ be the supremum of the accidentally writable ordinals.

Proposition (Hamkins, Lewis)

$\lambda < \zeta < \Sigma$.

Fact

If an ITTM can write a code for an ordinal α, then an ITTM can write a code for L_α.

Theorem (Welch)

The supremum of halting times of ITTMs on empty input equals λ.

We say that a triple (α, β, γ) is minimal with a property P if for any triple $(\alpha', \beta', \gamma')$ satisfying P, we have $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\gamma \leq \gamma'$.

Theorem (Welch)

(λ, ζ, Σ) *is minimal with* $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$.

ITTM-randomness

Definition (Hamkins, Lewis)

A set of reals A is semi-decidable if there is an ITTM M such that $M(x)\downarrow$ iff $x \in A$.

Membership of a real x in an ITTM-semidecidable set A is given by a recursion with λ^x stages – in analogy with Π^1_1 and ω_1^x.

Definition

A real z is ITTM-random if z is in no semi-decidable null set.

Definition

A real z is ITTM-decidable random if z is in no decidable null set.

ITTM-randomness

Proposition (Carl, S.)

The following are equivalent for a real z:

- *z is ITTM-decidable random.*
- *z is random over L_λ.*

Proof sketch.

Take a machine M deciding a measure 1 set not containing z. Then

$$\lambda(\{x \mid M(x) \downarrow = 0\}) = 1.$$

As the set of x with $\lambda^x = \lambda$ has measure 1, we have

$$\lambda(\{x \mid M(x)[\lambda] \downarrow = 0\}) = 1,$$

where $M(x)[\alpha]\downarrow$ means halting before α. Since λ is admissible, there is some $\alpha < \lambda$ with

$$\lambda(\{x \mid M(x)[\alpha] \downarrow = 0\}) = 1$$

The complement of this set contains z, is null and has a Borel code in L_λ. □

Theorem (Carl, S.)

The following are equivalent for a real z:

- z *is ITTM-random.*
- z *is random over* L_Σ *and* $\Sigma^z = \Sigma$.
- z *is random over* L_{λ^z}.

ITTM-randomness

Lemma (Carl, S.)

If $\Sigma^z > \Sigma$, then z is not ITTM-random.

Proof.

The set $\{z \mid \Sigma^z > \Sigma\}$ is ITTM-semidecidable, since we can search for a z-accidentally writable triple (α, β, γ) with $L_\alpha \prec_{\Sigma_1} L_\beta \prec_{\Sigma_2} L_\gamma$.

We will see below that this set is null. □

Lemma (Carl, S.)

If z is not random over L_Σ, then z is not ITTM-random.

Proof.

If z is not random over L_Σ, then we can search for an accidentally writable code for a Borel null set containing z. □

ITTM-randomness

Theorem (Carl, S.)

If z is random over $L_{\Sigma+1}$, then

$$L_\lambda[z] \prec_{\Sigma_1} L_\zeta[z] \prec_{\Sigma_2} L_\Sigma[z].$$

Hence $\lambda^z = \lambda$, $\zeta^z = \zeta$ and $\Sigma^z = \Sigma$.

Proof sketch.

Define Boolean values $[\![\varphi(\sigma)]\!]$ for Δ_0-formulas with parameters in L_Σ.

- $[\![\exists x \in \sigma\ \varphi(x, \sigma, \tau)]\!] = \bigvee_{(\nu,p)\in\sigma} [\![\varphi(\nu, \sigma, \tau)]\!] \wedge [p]$

Each $[\![\varphi(\sigma)]\!]$ is a Borel code in L_Σ.

The forcing theorem holds for Δ_0-formulas. □

Proof sketch.

Suppose that $\alpha < \beta$ are limit ordinals with $L_\alpha \prec_{\Sigma_2} L_\beta$ and z is random over some L_γ with β countable in L_γ. We claim that $L_\alpha[z] \prec_{\Sigma_2} L_\beta[z]$.

Suppose that $L_\beta \models \exists x\ \forall y\ \varphi(x,y,z)$, where $\varphi(x,y,z)$ is Δ_0.

The Boolean values of this statement and its reflection to α are

$$B_\beta = \bigcup_{x \in L_\beta} \bigcap_{y \in L_\beta} [\![\varphi(x,y,\dot{z})]\!]$$

$$B_\alpha = \bigcup_{x \in L_\alpha} \bigcap_{y \in L_\alpha} [\![\varphi(x,y,\dot{z})]\!].$$

We claim that $\lambda(B_\beta \setminus B_\alpha) = 0$. □

ITTM-randomness

Proof sketch.

We claim that $\lambda(B_\beta \setminus B_\alpha) = 0$. Let $\epsilon = \lambda(B_\beta)$. For every rational $\delta < \epsilon$,

$$(*)\ L_\beta \models \exists\mu\ \forall\nu\ \lambda(\bigcup_{x \in L_\mu} \bigcap_{y \in L_\nu} [\![\varphi(x, y, \dot{z})]\!]) > \delta.$$

Since $L_\alpha \prec_{\Sigma_2} L_\beta$, this holds in L_α for some μ.
Let

$$C = \bigcap_{\nu < \alpha} \bigcup_{x \in L_\mu} \bigcap_{y \in L_\nu} [\![\varphi(x, y, \dot{z})]\!].$$

By $(*)$, $\mu(C) \geq \delta$.

Let $D = \bigcup_{x \in L_\mu} \bigcap_{y \in L_\alpha} [\![\varphi(x, y, \dot{z})]\!] \subseteq B_\alpha$. We claim that $\lambda(D) \geq \delta$. Since this holds for all $\delta < \epsilon$, we have $\lambda(B_\alpha) \geq \epsilon$ and $\lambda(B_\alpha \triangle B_\beta) = 0$.

Assume $\lambda(C \setminus D) > 0$ and take $z \in C \setminus D$. We can assume that α is z-admissible, since this holds for a measure 1 set of z.

For every $x \in L_\mu$, there is some $\nu < \alpha$ and $y \in L_\nu$ with $z \notin [\![\varphi(x, y, \dot{z})]\!]$. We can fix ν by admissibility. This contradicts $z \in C$. □

Angles D'Auriac and Monin (2018) analyzed these randomness notions – in particular, the relationship with α-ML-randomness for $\alpha \in \{\lambda, \zeta, \Sigma\}$.

Definition

A real is α-ML-random if it avoids every $\mathbf{\Pi}^0_2$ set $\bigcap_n U_n$ with $\lambda(U_n) \leq 2^{-n}$ that has a code which is Σ_1-definable over L_α.

Their main open question:

Question

Is there a random real over L_Σ that is not ITTM-random?

The difference to ω_1^{ck}

Fact

Assume that there is a $\mathbf{\Pi}_2^0$ set A such that

- *The reals of L_Σ are contained in A and*
- *A contains no ITTM-random reals.*

Then every sufficiently class-generic real for random forcing over L_Σ fails to be ITTM-random.

This fact follows from Yu's proof of $\omega_1^x > \omega_1^{ck}$.

However, there is no such set $A = \bigcap_n U_n$ with $\vec{U} = \langle U_n \mid n \in \omega \rangle$ uniformly ITTM-semidecidable.

Characterizations

Definition (Angles D'Auriac, Monin)

A set of reals A is Σ_n^α if $A = \{x \mid L_\alpha[x] \models \varphi(x)\}$ for some Σ_n-formula $\varphi(x)$.

Theorem (Angles D'Auriac, Monin)

The following are equivalent:

- *z is ITTM-random*
- *z avoids every Δ_3^Σ null set,[1] with parameters in L_ζ.*

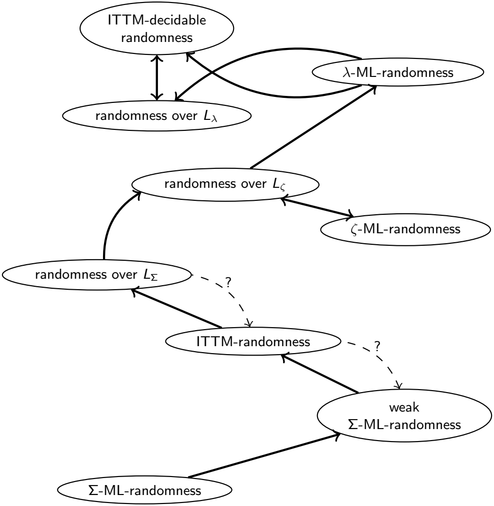
Theorem (Angles D'Auriac, Monin)

The following are equivalent:

- *z is random over L_Σ.*
- *z avoids every Σ_2^Σ null set, with parameters in L_ζ.*
- *z avoids every Π_2^Σ null set, with parameters in L_ζ.*

[1] whose definition is compatible with both ζ and Σ

Diagram of randomness notions



Literature

[1] Carl, Schlicht: Randomness via infinite computation and effective descriptive set theory, JSL 2018

[2] Angles D'Auriac, Monin: Genericity and randomness with ITTMs, submitted