MM^{++} implies $(*)$

David Asperó* and Ralf Schindler†

June 17, 2019

Abstract

We show that Martin's Maximum^{++} implies Woodin's $\mathbb{P}_{\max}$ axiom $(*)$.

1 Introduction.

Cantor's Continuum Problem, which later became Hilbert's first Problem (see [7]), asks how many real numbers there are. This question got a non-answer through the discovery of the method of *forcing* by Paul Cohen: CH, Cantor's Continuum Hypothesis, is independent from ZFC (see [3]), the standard axiom system for set theory which had been isolated by Zermelo and Fraenkel. CH states that every uncountable set of reals has the same size as $\mathbb{R}$.

Ever since Cohen, set theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem. See e.g. [21], [22], [9], and the discussion in [4]. There are two prominent such axioms which decide CH in the negative and which in fact both prove that there are $\aleph_2$ reals: Martin's Maximum (MM, for short) or variants thereof on the one hand (see [6]), and Woodin's axiom $(*)$ on the other hand (see [20]). See e.g. [12], [19], and [14].

Both of these axioms may be construed as maximality principles for the theory of the structure $(H_{\omega_2}; \in)$, but up to this point the relationship between MM and $(*)$ was a bit of a mystery, which led M. Magidor to call $(*)$ a "competitor" of MM ([12, p. 18]). Both MM and $(*)$ are inspired by and formulated in the language of forcing,

*Funded by EPSRC Grant EP/N032160/1.

†Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germanys Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

and they both have "the same intuitive motivation: Namely, the universe of sets is rich" ([12, p. 18]).

This paper resolves the tension between MM and $(*)$ by proving that MM^{++}, a strengthening of MM, actually *implies* $(*)$, see Theorem 2.1 below, so that MM and $(*)$ are actually compatible with each other. This answers [20, Question (18) a) on p. 924], see also [20, p. 846], [12, Conjecture 6.8 on p. 19], and [14, Problem 14.7].

2 Preliminaries.

Martin's Maximum^{++}, abbreviated by MM^{++}, see [6] (cf. also [20, Definition 2.45 (2)]), is the statement that if $\mathbb{P}$ is a forcing which preserves stationary subsets of ω_1, if $\{D_i : i < \omega_1\}$ is a collection of dense subsets of $\mathbb{P}$, and if $\{\tau_i : i < \omega_1\}$ is a collection of $\mathbb{P}$-names for stationary subsets of ω_1, then there is a filter $g \subset \mathbb{P}$ such that for every $i < \omega_1$,

(i) $g \cap D_i \neq \emptyset$ and

(ii) $(\tau_i)^g = \{\xi < \omega_1 : \exists p \in g\, p \Vdash_{\mathbb{P}} \check{\xi} \in \tau_i\}$ is stationary.

Woodin's $\mathbb{P}_{\max}$ axiom (*), see [20, Definition 5.1], is the statement that

(i) AD holds in $L(\mathbb{R})$ and

(ii) there is some g which is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \cap V \subset L(\mathbb{R})[g]$.

Already PFA, the *Proper Forcing Axiom*, which is weaker than MM^{++}, implies $\mathsf{AD}^{L(\mathbb{R})}$ and much more, see [17], [8], and [15, Chapter 12]. This paper produces a proof of the following result.

Theorem 2.1 *Assume* Martin's Maximum^{++}*. Then Woodin's* $\mathbb{P}_{\max}$*-axiom* $(*)$ *holds true.*

Our key new idea is $(\Sigma.8)$ on page 9 below.

P. Larson, see [10] and [11], has shown that $\mathsf{MM}^{+\omega}$ is consistent with $\neg(*)$ relative to a supercompact limit of supercompact cardinals.

Throughout our entire paper, "ω_1" will *always* denote ω_1^V, the ω_1 of V.

Let us fix throughout this paper some $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$. Let us define g_A as the set of all $\mathbb{P}_{\max}$ conditions $p = (N; \in, I, a)$ such that there is a generic iteration

$$(N_i, \sigma_{ij} : i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$,[1] then $I^* = (\mathsf{NS}_{\omega_1})^V \cap N_{\omega_1}$ and $a^* = A$.

Lemma 2.2 (Woodin) *Assume that NS_{ω_1} is saturated and that $\mathcal{P}(\omega_1)^{\#}$ exists.*

(1) g_A is a filter.

(2) If g_A is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, then $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

Proof. This routinely follows from the proof of [20, Lemma 3.12 and Corollary 3.13] and from [20, Lemma 3.10]. □

One may also use BMM plus "NS_{ω_1} is precipitous" to show that g_A is a filter, this is by the proof from [2].

Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. We say that Γ is *productive* iff for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$, if D is universally Baire (see [5]) as being witnessed by the trees T and U on ${}^{k+2}\omega \times \mathrm{OR}$, i.e., $D = p[T]$ and for all posets $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \; p[U] = \mathbb{R}^{k+2} \setminus p[T], \tag{1}$$

and if

$$\tilde{U} = \{(s \restriction (k+1), (s(k+1), t)) \colon (s,t) \in U\}, \tag{2}$$

so that $(x_0, \ldots, x_k) \in p[\tilde{U}]$ iff there is some y such that $(x_0, \ldots, x_k, y) \in p[U]$, then there is a tree $\tilde{T}$ on ${}^{k+1}\omega \times \mathrm{OR}$ such that for all posets $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \; p[\tilde{U}] = \mathbb{R}^{k+1} \setminus p[\tilde{T}]. \tag{3}$$

Let us denote by Γ^∞ the collection of all $D \in \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$ which are universally Baire. If $D \in \Gamma^\infty$, then there is an unambiguous version of D in any forcing extension of V, which as usual we denote by D^*. (2) then means that if $D = p[U]$ and $E = p[\tilde{U}]$, then in any forcing extension of V, $E^* = \exists^{\mathbb{R}} D^*$.

If $\Gamma \subset \Gamma^\infty$ is productive and if $D \in \Gamma$, then any projective statement about D is absolute between V and any forcing extension of V,[2] i.e., if φ is projective, $x_1, \ldots, x_k \in \mathbb{R}$, and $\mathbb{P}$ is any poset, then

$$V \models \varphi(x_1, \ldots, x_k, D) \iff \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \ldots, \check{x}_k, D^*).$$

[1] Here and elsewhere we often confuse a model with its underlying universe.

[2] This seems to be wrong if we just assume $\Gamma \subset \Gamma^\infty$, but the hypothesis that Γ be productive is crossed out.

By a theorem of Woodin, see e.g. [18, Theorem 1.2], combined with the key result of Martin and Steel in [13], the pointclass Γ^∞ is productive under the hypothesis that there is a proper class of Woodin cardinals.

The proof from [17] produces the result that under PFA, the universe is closed under the operation $X \mapsto M_\omega^\#(X)$, which implies that every set of reals in $L(\mathbb{R})$ is universally Baire and that $\bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R})$ is productive. See e.g. [16, Section 3, pp. 187f.] on the relevant argument. Therefore, in the light of Lemma 2.2, Theorem 2.1 follows from the following more general statement.

Theorem 2.3 *Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that*

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) Γ *is productive, and*

(iv) Martin's Maximum^{++} *holds true.*

Then g_A is $\mathbb{P}_{\max}$-generic over $L(\Gamma, \mathbb{R})$.[3]

In the light of Lemma 2.2 and [6, Corollary 17], Theorem 2.3 readily follows from the following via a standard application of MM^{++}.

Lemma 2.4 *Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that*

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) Γ *is productive, and*

(iv) NS_{ω_1} *is saturated.*[4]

Let $D \subset \mathbb{P}_{\max}$ be open dense, $D \in L(\Gamma, \mathbb{R})$.[5] *There is then a stationary set preserving forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there is some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration*

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A$.

[3] In the presence of a proper class of Woodin cardinals, hypotheses (i), (ii), and (iv) then give $(*)_\Gamma$, see [16, Definition 4.1].

[4] We could weaken this hypothesis to "NS_{ω_1} is precipitous."

[5] By hypothesis, D is then universally Baire in the codes, so that there is an unambiguous version of D in any forcing extension of V, which again we denote by D^*.

The attentive reader will notice that we don't need the full power of MM^{++} in order to derive Theorem 2.3 from Lemma 2.4, the hypothesis that D-BMM^{++} holds true for all $D \in \Gamma^\infty$ would suffice, see [20, Definition 10.123]. By the proof of [1, Theorem 2.7], $(*)$ is then actually *equivalent* to a version of BMM; we state this as follows.

Theorem 2.5 *Let* $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. *Assume that*

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) Γ *is productive, and*

(iv) NS_{ω_1} *is saturated.*[6]

The following statements are then equivalent.

(1) D-BMM^{++} *holds true for all* $D \in \Gamma$.

(2) g_A *is* $\mathbb{P}_{\max}$*-generic over* $L(\Gamma, \mathbb{R})$.

Theorem 2.6 *Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.*

(1) D-BMM^{++} *holds true for all* $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) $(*)$.

Our next section is entirely devoted to a proof of Lemma 2.4.

The authors thank Andreas Lietz for his comments on earlier versions of this paper.

3 The forcing.

Let us assume throughout the hypotheses of Lemma 2.4. We aim to verify its conclusion.

Let us fix $D \subset \mathbb{P}_{\max}$, an open dense set in $L(\Gamma, \mathbb{R})$. By hypotheses (ii) and (iii) in the statement of Lemma 2.4 we will have that

[6] Again, we could weaken this hypothesis to "NS_{ω_1} is precipitous."

(D.1) $V^{\mathrm{Col}(\omega,\omega_2)} \models$ "D^* is an open dense subset of $\mathbb{P}_{\max}$."

Let us identify D with a canonical set of reals coding the elements of D,[7] and let $T \in V$ be a tree on $\omega \times 2^{\aleph_2}$ such that

(D.2) $V^{\mathrm{Col}(\omega,\omega_2)} \models D^* = p[T]$.

Let us write

$$\kappa = (2^{\aleph_2})^+, \tag{4}$$

so that $T \in H_\kappa$. Let d be $\mathrm{Col}(\kappa,\kappa)$-generic over V. In $V[d]$, let $(\bar{A}_\lambda \colon \lambda < \kappa)$ be a $\diamondsuit_\kappa$-sequence, i.e., for all $\bar{A} \subset \kappa$, $\{\lambda < \kappa \colon \bar{A} \cap \lambda = \bar{A}_\lambda\}$ is stationary. Also, let $c \colon \kappa \to H_\kappa^V = H_\kappa^{V[d]}$, $c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$$Q_\lambda = c\text{"}\lambda \text{ and } A_\lambda = c\text{"}\bar{A}_\lambda. \tag{5}$$

Let $C \subset \kappa$ be club such that for all $\lambda \in C$,

(i) Q_λ is transitive,

(ii) $\{T, ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)\} \cup 2^{\aleph_2} \subset Q_\lambda$,

(iii) $Q_\lambda \cap \mathrm{OR} = \lambda$ (so that $c \upharpoonright \lambda \colon \lambda \to Q_\lambda$ is bijective), and

(iv) $(\mathcal{Q}_\lambda; \in) \prec (H_\kappa; \in)$.

In $V[d]$, for all P, $B \subset H_\kappa$, the set of all $\lambda \in C$ such that

$$(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)$$

is club, and the set of all $\lambda \in C$ such that $B \cap Q_\lambda = A_\lambda$ is stationary, so that

($\diamondsuit$) For all P, $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B)\}$$

is stationary.

[7] We will have to spell out a bit more precisely below in which way we aim to have the elements of $p[T]$ code the elements of D, see (Σ.5) below.

We shall sometimes also write $Q_\kappa = H_\kappa$.

We shall now go ahead and produce a stationary set preserving forcing $\mathbb{P} \in V[d]$ which adds some $p \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V[d]^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A$. As the forcing $\mathrm{Col}(\kappa,\kappa)$ which added d is certainly stationary set preserving, this will verify Lemma 2.4.

NS_{ω_1} is still saturated in $V[d]$ and (D.1) and (D.2) are still true in $V[d]$, so that in order to simplify our notation, we shall in what follows confuse $V[d]$ with V, i.e., pretend that in addition to "NS_{ω_1} is saturated" plus (D.1) and (D.2), ($\Diamond$) is also true in V.

Working under these hypotheses, we shall now recursively define a $\subset$-increasing and continuous chain of forcings $\mathbb{P}_\lambda$ for all $\lambda \in C \cup \{\kappa\}$. The forcing $\mathbb{P}$ will be $\mathbb{P}_\kappa$.

Assume that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}_\mu$ has already been defined in such a way that $\mathbb{P}_\mu \subset Q_\mu$ for all $\mu \in C \cap \lambda$.

We shall be interested in objects $\mathfrak{C}$ which exist in some outer model[8] and which have the following properties.

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle, \tag{6}$$

and

(C.1) M_0, $N_0 \in \mathbb{P}_{\max}$,

(C.2) $x = \langle k_n \colon n < \omega \rangle$ is a real code for N_0 and $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$,

(C.3) $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0} \rangle \in N_0$ is a generic iteration of M_0 which witnesses that $N_0 < M_0$ in $\mathbb{P}_{\max}$,

(C.4) $\langle N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle$ is a generic iteration of N_0 such that if

$$N_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*),$$

then $A^* = A$,[9]

[8] W is an outer model iff W is a transitive model of ZFC with $W \supset V$ and which has the same ordinals as V; in other words, W is an outer model iff V is an inner model of W.

[9] There is no requirement on I^* matching the non-stationary ideal of some model in which $\mathfrak{C}$ exists.

(C.5) $\langle M_i, \pi_{ij} \colon i \le j \le \omega_1 \rangle = \sigma_{0\omega_1}(\langle M_i, \pi_{ij} \colon i \le j \le \omega_1^{N_0} \rangle)$ and

$$M_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A), \tag{7}$$

(C.6) $K \subset \omega_1$,

and for all $\delta \in K$,

(C.7) $\lambda_\delta < \lambda$, and if $\gamma < \delta$ is in K, then $\lambda_\gamma < \lambda_\delta$ and $X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta$, and

(C.8) $X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$ and $X_\delta \cap \omega_1 = \delta$.

We need to define a language $\mathcal{L}$ (independently from λ) whose formulae will be able to describe $\mathfrak{C}$ with the above properties by producing the models M_i and N_i, $i < \omega_1$, as term models out of equivalence classes of terms of the form $\dot{n}$, $n < \omega$. The language $\mathcal{L}$ will have the the following constants.

$\dot{T}$	intended to denote T
x for every $x \in H_\kappa$	intended to denote x itself
$\dot{n}$ for every $n < \omega$	as terms for elements of M_i and $N_i, i < \omega_1$
$\dot{M}_i$ for $i < \omega_1$	intended to denote M_i
$\dot{\pi}_{ij}$ for $i \le j \le \omega_1$	intended to denote π_{ij}
$\dot{\vec{M}}$	intended to denote $(M_j, \pi_{jj'} \colon j \le j' \le \omega_1^{N_i})$ for $i < \omega_1$
$\dot{N}_i$ for $i < \omega_1$	intended to denote N_i
$\dot{\sigma}_{ij}$ for $i \le j < \omega_1$	intended to denote σ_{ij}
$\dot{a}$	intended to denote the distinguished a-predicate of $M_i, N_i, i < \omega_1$
$\dot{I}$	intended to denote the distinguished ideal of $N_i, i < \omega_1$
$\dot{X}_\delta$ for $\delta < \omega_1$	intended to denote X_δ.

Formulae of $\mathcal{L}$ will be of the following form.

$$\ulcorner \dot{N}_i \vDash \varphi(\xi_1, \ldots, \xi_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_1 r_1}, \ldots, \dot{\pi}_{q_s r_s}, \dot{\vec{M}}) \urcorner$$

for $i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \le r_1 < \omega_1, \ldots, q_s \le r_s < \omega_1$

$$\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = x \urcorner \quad \text{for } i < \omega_1 \text{ and } x \in H_{\omega_2}$$

$$\ulcorner \dot{\pi}_{\omega_1\omega_1}(x) = x \urcorner \quad \text{for } x \in H_{\omega_2}$$

$$\ulcorner \dot{\sigma}_{ij}(\dot{n}) = \dot{m} \urcorner \quad \text{for } i \le j < \omega_1, n, m < \omega$$

$$\ulcorner(\vec{u},\vec{\alpha})\in\dot{T}\urcorner \quad \text{for } \vec{u}\in{}^{<\omega}\omega \text{ and } \vec{\alpha}\in{}^{<\omega}(2^{\aleph_2})$$
$$\ulcorner\delta\mapsto\bar{\lambda}\urcorner \quad \text{for } \delta<\omega_1, \bar{\lambda}<\kappa$$
$$\ulcorner x\in\dot{X}_\delta\urcorner \quad \text{for } \delta<\omega_1, x\in H_\kappa$$

Let us write $\mathcal{L}^\lambda$ for the collection of all $\mathcal{L}$-formulae except for the formulae which mention elements outside of Q_λ, i.e., except for the formulae of the form $\ulcorner\delta\mapsto\bar{\lambda}\urcorner$ for $\delta<\omega_1$ and $\lambda\le\bar{\lambda}<\kappa$ and $\ulcorner x\in\dot{X}_\delta\urcorner$ for $\delta<\omega_1$ and $x\in H_\kappa\setminus Q_\lambda$. We may and shall assume that $\mathcal{L}$ is built in a canonical way so that $\mathcal{L}^\lambda\subset Q_\lambda$.

We say that the objects $\mathfrak{C}$ as in (6) are *pre-certified* by a collection Σ of $\mathcal{L}^\lambda$-formulae if and only if (C.1) through (C.8) are satisfied by $\mathfrak{C}$ and there are surjections $e_i\colon\omega\to N_i$ for $i<\omega_1$ such that the following hold true.

(Σ.1) $\ulcorner\dot{N}_i\vDash\varphi(\xi_1,\dots,\xi_k,\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I},\dot{M}_{j_1},\dots,\dot{M}_{j_m},\dot{\pi}_{q_1r_1},\dots,\dot{\pi}_{q_sr_s},\dot{\vec{M}})\urcorner\in\Sigma$ iff $i<\omega_1$, ξ_1, …, $\xi_k\le\omega_1^{N_i}$, n_1, … , $n_\ell<\omega$, j_1, …, $j_m\le\omega_1^{N_i}$, $q_1\le r_1\le\omega_1^{N_i}$, …, $q_s\le r_s\le\omega_1^{N_i}$ and
$N_i\vDash\varphi(\xi_1,\dots,\xi_k,e_i(n_1),\dots,e_i(n_\ell),A\cap\omega_1^{N_i},I^{N_i},M_{j_1},\dots,M_{j_m},\pi_{q_1r_1},\dots,\pi_{q_sr_s},\vec{M})$, where I^{N_i} is the distinguished ideal of N_i and $\vec{M}=\langle M_j,\pi_{jj'}\colon j\le j'\le\omega_1^{N_i}\rangle$,

(Σ.2) $\ulcorner\dot{\pi}_{i\omega_1}(\dot{n})=x\urcorner\in\Sigma$ iff $i<\omega_1$, $n<\omega$, and $\pi_{i\omega_1}(e_i(n))=x$,

(Σ.3) $\ulcorner\dot{\pi}_{\omega_1\omega_1}(x)=x\urcorner\in\Sigma$ iff $x\in H_{\omega_2}$,

(Σ.4) $\ulcorner\dot{\sigma}_{ij}(\dot{n})=\dot{m}\urcorner\in\Sigma$ iff $i\le j<\omega_1$, n, $m<\omega$, and $\sigma_{ij}(e_i(n))=e_j(m)$,

(Σ.5) letting F with $\mathrm{dom}(F)=\omega$ be the monotone enumeration of the Gödel numbers of all $\ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner$ with $\ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner\in\Sigma$, we have that $\ulcorner(\vec{u},\vec{\alpha})\in\dot{T}\urcorner\in\Sigma$ iff there is some $n<\omega$ such that $\langle\vec{u},\vec{\alpha}\rangle=\langle(F(m),\alpha_m)\colon m<n\rangle$ and $F(m)=k_m$ for all $m<n$,

(Σ.6) $\ulcorner\delta\mapsto\bar{\lambda}\urcorner\in\Sigma$ iff $\delta\in K$ and $\bar{\lambda}=\lambda_\delta$, and

(Σ.7) $\ulcorner x\in\dot{X}_\delta\urcorner\in\Sigma$ iff $\delta\in K$ and $x\in X_\delta$.

We say that the objects $\mathfrak{C}$ as in (6) are *certified* by a collection Σ of formulae if and only if $\mathfrak{C}$ is pre-certified by Σ and in addition,

(Σ.8) if $\delta\in K$, then $[\Sigma]^{<\omega}\cap X_\delta\cap E\neq\emptyset$ for every $E\subset\mathbb{P}_{\lambda_\delta}$ which is dense in $\mathbb{P}_{\lambda_\delta}$ and definable over the structure

$$(Q_{\lambda_\delta};\in,\mathbb{P}_{\lambda_\delta},A_{\lambda_\delta})$$

from parameters in X_δ.[10]

By way of definition, we call $\mathfrak{C}$ as in (6) a *semantic certificate* iff there is a collection Σ of formulae such that $\mathfrak{C}$ is certified by Σ. We call Σ a *syntactic certificate* iff there is a semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ. Given a syntactic certificate Σ, there is a *unique* semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ. Even though it is obvious how to construct $\mathfrak{C}$ from Σ, in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given Σ.

Let $\Sigma \cup p$ be a set of formulae, where p is finite. We say that p is *certified by* Σ if and only if there is some (unique) $\mathfrak{C}$ as in (6) such that $\mathfrak{C}$ is certified by Σ and

(Σ.9) $p \in [\Sigma]^{<\omega}$.

We may also say that p is *certified by* $\mathfrak{C}$ as in (6) iff there is some Σ such that $\mathfrak{C}$ and p are both certified by Σ – and we will then also refer to Σ as a syntactical certificate for p and to $\mathfrak{C}$ as the associated semantical certificate.

We are then ready to define the forcing $\mathbb{P}_\lambda$. We say that $p \in \mathbb{P}_\lambda$ if and only if

$$V^{\mathrm{Col}(\omega,\lambda)} \models \text{“There is a set } \Sigma \text{ of } \mathcal{L}^\lambda\text{-formulae such that } p \text{ is certified by } \Sigma.\text{”} \tag{8}$$

Let p be a finite set of formulae of $\mathcal{L}^\lambda$. By the homogeneity of $\mathrm{Col}(\omega,\lambda)$, if there is some h which is $\mathrm{Col}(\omega,\lambda)$-generic over V and there is some $\Sigma \in V[h]$ such that p is certified by Σ, then for all h which are $\mathrm{Col}(\omega,\lambda)$-generic over V there is some $\Sigma \in V[h]$ such that p is certified by Σ. It is then easy to see that $\langle \mathbb{P}_\lambda \colon \lambda \in C \cup \{\kappa\}\rangle$ is definable over V from $\langle A_\lambda \colon \lambda < \kappa\rangle$ and C, and is hence an element of V.

Again let p be a finite set of formulae of $\mathcal{L}^\lambda$. By Σ^1_1 absoluteness, if there is any outer model in which there is some Σ which certifies p, then there is some $\Sigma \in V^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.[11] This simple observation is important in the verification that $\mathbb{P}_\lambda$ is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

(i) $\mathbb{P} = \mathbb{P}_\kappa \subset H_\kappa$,

(ii) if $\bar\lambda < \lambda$ are both in $C \cup \{\kappa\}$, then $\mathbb{P}_{\bar\lambda} \subset \mathbb{P}_\lambda$, and

[10] Equivalently, $[\Sigma]^{<\omega} \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ which is dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$ and definable over the structure

$$(X_\delta; \in, \mathbb{P}_{\lambda_\delta} \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)$$

from parameters in X_δ.

[11] In fact, if P is a transitive model of KP plus the axiom *Beta* with $(Q_\lambda; \langle A_{\bar\lambda} \colon \bar\lambda < \lambda\rangle) \in P$ and if $p \in \mathbb{P}_\lambda$, then there is some $\Sigma \in P^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.

(iii) if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $\mathbb{P}_\lambda = \bigcup_{\bar{\lambda} \in C \cap \lambda} \mathbb{P}_{\bar{\lambda}}$,

so that there is some club $D \subset C$ such that for all $\lambda \in D$,

$$\mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda.$$

Hence ($\diamondsuit$) gives us the following.

($\diamondsuit(\mathbb{P})$) For all $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\}$$

is stationary.

The first one of the following lemmas is entirely trivial.

Lemma 3.1 *Let Σ be a syntactic certificate, and let p, $q \in [\Sigma]^{<\omega}$. Then p and q are compatible conditions in $\mathbb{P}$.*

Lemma 3.2 *Let $\lambda \in C \cup \{\kappa\}$. Then $\emptyset \in \mathbb{P}_\lambda$.*

Proof. See the proof of [1, Theorem 2.8], or the proof of [16, Theorem 4.2]. Notice that for all $\lambda \in C \cup \{\kappa\}$, $\emptyset \in \mathbb{P}_\lambda$ iff $\emptyset \in \mathbb{P}$.

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V, and write $\rho = \omega_3^V = \omega_1^{V[h]}$. Inside $V[h]$,

$$((H_{\omega_2})^V; \in (\mathsf{NS}_{\omega_1})^V, A)$$

is a $\mathbb{P}_{\max}$ condition, call it p. Let $q \in (\mathbb{P}_{\max})^{V[h]}$, $q < p$, $q \in D^*$, cf. (D.1). Let $q = N_0 = (N_0; \in, I, a)$. Let $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0}) \in N_0$ be the unique generic iteration of p which witnesses $q < p$.

Let $(N_i, \sigma_{ij} \colon i \leq j \leq \rho) \in V[h]$ be a generic iteration of N_0 such that $\rho = \omega_1^{N_\rho}$.[12] Let

$$(M_i, \pi_{ij} \colon i \leq j \leq \rho) = \sigma_{0\rho}((M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0})) \tag{9}$$

We may lift (10) to a generic iteration

$$(M_i^+, \pi_{ij}^+ \colon i \leq j \leq \rho) \tag{10}$$

of V. Let us write $M = M_\rho^+$ and $\pi = \pi_{0\rho}^+$.

[12]If we wished, then we could even arrange that writing $N_\rho = (N_\rho; \in, I^*, a^*)$, we have that $I^* = (\mathsf{NS}_\rho)^{V[h]} \cap N_\rho$, but this is not relevant here; cf. footnote 9.

Let $\langle k_n, \alpha_n \colon n < \omega \rangle$ be such that $x = \langle k_n \colon n < \omega \rangle$ is a real code for N_0 à la (Σ.5) and $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$. We then clearly have that $\langle (k_n, \pi(\alpha_n)) \colon n < \omega \rangle \in [\pi(T)]$.

It is now easy to see that

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \rho \rangle, \langle (k_n, \pi(\alpha_n)) \colon n < \omega \rangle, \langle\rangle \tag{11}$$

certifies $\emptyset$, construed as the empty set of $\pi(\mathcal{L}^\kappa)$ formulae: as the third component $\langle\rangle$ of $\mathfrak{C}$ in (11) is empty, *any* set of surjections $e_i \colon \omega \to N_i$, $i < \omega_1$, will induce a syntactic certificate for $\emptyset$ whose associated sematic certificate is $\mathfrak{C}$. By Σ^1_1 absoluteness, there is then some $\mathfrak{C} \in M^{\mathrm{Col}(\omega, \pi(\omega_2))}$ as in (11) which certifies $\emptyset$ so that $\emptyset \in \pi(\mathbb{P})$. By the elementarity of π, then, there is some $\mathfrak{C} \in V^{\mathrm{Col}(\omega,\omega_2)}$ which certifies $\emptyset$, construed as the empty set of $\mathcal{L}^\kappa$ formulae. Hence $\emptyset \in \mathbb{P}$. □

Lemma 3.3 *Let $\lambda \in C \cup \{\kappa\}$. Let $g \subset \mathbb{P}_\lambda$ be a filter such that $g \cap E \neq \emptyset$ for all dense $E \subset \mathbb{P}_\lambda$ which are definable over $(Q_\lambda; \in, \mathbb{P}_\lambda)$ from elements of Q_λ. Then $\bigcup g$ is a syntactic certificate.*

Proof. It is obvious how to read off from $\bigcup g$ a candidate

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle$$

like in (6) for a semantical certificate for $\bigcup g$. Let us be somewhat explicit, though. A variant of what is to come shows how to derive $\mathfrak{C}$ from a given syntactic certificate Σ, where $\mathfrak{C}$ is unique such that Σ certifies $\mathfrak{C}$, cf. the remark on p. 10.

For i, $j < \omega_1$ and τ, $\sigma \in \{\dot{n} \colon n < \omega\} \cup \omega_1$ define

$$\begin{aligned}
\tau \sim_i \sigma \quad & \text{iff} \quad \ulcorner \dot{N}_i \models \tau = \sigma \urcorner \in \bigcup g \\
(i,\tau) \sim_{\omega_1} (j,\sigma) \quad & \text{iff } i \leq j \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ij}(\tau) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho = \sigma \urcorner\} \subset \bigcup g \\
& \quad \text{or } j \leq i \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ji}(\sigma) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho = \tau \urcorner\} \subset \bigcup g \\
[\tau]_i &= \{\sigma \colon \tau \sim_i \sigma\} \\
[(i,\tau)] &= \{(j,\sigma) \colon (i,\tau) \sim_{\omega_1} (j,\sigma)\} \\
M_i &= \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \omega_1 \wedge \ulcorner \dot{N}_i \models \tau \in \dot{M}_i \urcorner \in \bigcup g\} \\
M_{\omega_1} &= (H_{\omega_2})^V \\
N_i &= \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \omega_1\} \\
N_{\omega_1} &= \{[i,\tau]_{\omega_1} \colon i < \omega_1 \wedge \tau \in \dot{M}_i \urcorner \in \bigcup g\}
\end{aligned}$$

$$
\begin{aligned}
[\tau]_i \,\tilde{\in}_i\, [\sigma]_i \quad & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \sigma \urcorner \in \bigcup g \\
[i,\tau] \tilde{\in}_{\omega_1} [j,\sigma] \quad & \text{iff } i \le j \wedge \exists \rho\, \{\ulcorner \dot{\sigma}_{ij}(\tau) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho \in \sigma \urcorner\} \subset \bigcup g \\
& \quad \text{or } j \le i \wedge \exists \rho\, \{\ulcorner \dot{\sigma}_{ji}(\sigma) = \rho \urcorner, \ulcorner \dot{N}_j \models \tau \in \rho \urcorner\} \subset \bigcup g \\
[\tau]_i \in I^{N_i} \quad & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \dot{I} \urcorner \in \bigcup g \\
[i,\tau] \in I^{N_{\omega_1}} \quad & \text{iff} \quad [\tau]_i \in I^{N_i} \\
[\tau]_i \in a^{N_i} \quad & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \dot{a} \urcorner \in \bigcup g \\
[i,\tau] \in a^{N_{\omega_1}} \quad & \text{iff} \quad [\tau]_i \in I^{N_i} \\
\pi_{ij}([\tau]_i) = [\sigma]_j \quad & \text{iff} \quad \ulcorner \dot{N}_j \models \dot{\pi}_{ij}(\tau) = \sigma \urcorner \in \bigcup g \\
\pi_{i\omega_1}([\tau]_i) = x \quad & \text{iff} \quad \ulcorner \dot{\pi}_{i\omega_1}(\tau) = x \urcorner \in \bigcup g \\
\pi_{\omega_1\omega_1}(x) = x \quad & \text{iff} \quad x \in (H_{\omega_2})^V \\
\sigma_{ij}([\tau]_i) = [\sigma]_j \quad & \text{iff} \quad \ulcorner \dot{\sigma}_{ij}(\tau) = \sigma \urcorner \in \bigcup g \\
\sigma_{i\omega_1}([\tau]_i) = [i,\tau] \quad & \\
(k,\alpha) = (k^g_n, \alpha^g_n) \quad & \text{iff } \exists \vec{u} \exists \vec{\alpha} (\ulcorner (\vec{u},\vec{\alpha}) \in \dot{T} \urcorner \in \bigcup g \wedge k = \vec{u}(n) \wedge \alpha = \vec{\alpha}(n)) \\
\delta \in K^g \quad & \text{iff } \exists \bar{\lambda} \ulcorner \delta \mapsto \bar{\lambda} \urcorner \in \bigcup g \\
\bar{\lambda} = \lambda^g_\delta \quad & \text{iff } \delta \in K^g \wedge \ulcorner \delta \mapsto \bar{\lambda} \urcorner \in \bigcup g \\
x \in X^g_\delta \quad & \text{iff } \delta \in K^g \wedge \ulcorner x \in \dot{X}_\delta \urcorner \in \bigcup g
\end{aligned}
$$

We will first have that $\tilde{\in}_0$ is wellfounded and that in fact (the transitive collapse of) the structure $N_0 = (N_0; \tilde{\in}_0, a^{N_0}, I^{N_0})$ is an iterable $\mathbb{P}_{\max}$ condition. This is true because straightforward density arguments give (C.2), i.e., that $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$ and $\langle k_n \colon n < \omega \rangle$ will code the theory of N_0 à la (Σ.5).

Another set of easy density arguments will give that $(N_i, \sigma_{ij} \colon i \le j \le \omega_1)$ is a generic iteration of N_0, were we identify N_i with the structure $(N_i; \tilde{\in}_i, a^{N_i}, I^{N_i})$. To verify this, let us first show:

Claim 3.4 *For each $i < \omega_1$ and for each $\xi \le \omega_1^{N_i}$, $[\xi]_i$ represents ξ in (the transitive collapse of the well-founded part of) the term model for N_i; moreover, $a^{N_i} = A \cap \omega_1^{N_i}$. Hence $a^{N_{\omega_1}} = A$.*

Proof of Claim 3.4. Straightforward arguments using (Σ.1) show that $[\xi]_i$ must always represent ξ in N_i as given by any certificate. Claim 3.4 then follows by straightforward density arguments. □ (Claim 3.4)

Similarily:

Claim 3.5 *Let $i < \omega_1$. N_{i+1} is generated from* $\mathrm{ran}(\sigma_{ii+1}) \cup \{\omega_1^{N_i}\}$ *in the sense that for every $x \in N_{i+1}$ there is some function $f \in {}^{\omega_1^{N_i}}(N_i) \cap N_i$ such that $x = \sigma_{ii+1}(f)(\omega_1^{N_i})$.*

Claim 3.6 *Let $i < \omega_1$. $\{X \in \mathcal{P}(\omega_1^{N_i}) \cap N_i \colon \omega_1^{N_i} \in \sigma_{ii+1}(X)\}$ is generic over N_i for the forcing given by the I^{N_i}-positive sets.*

Claim 3.7 *Let $i \leq \omega_1$ be a limit ordinal. For every $x \in N_i$ there is some $j < i$ and some $\bar{x} \in N_j$ such that $x = \sigma_{ji}(\bar{x})$.*

$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$ is then indeed a generic iteration of N_0. As N_0 is iterable, we may and shall identify N_i with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that $\mathfrak{C}$ satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify M_i with the structure $(M_i; \in, (\mathsf{NS}_{\omega_1^{M_i}})^{M_i}, A \cap \omega_1^{M_i})$. Let us now verify (C.8) and (C.9).

As for (C.8), $X_\delta \cap \omega_1 = \delta$ for $\delta \in K$ is easy. Let $x_1, \ldots, x_k \in X_\delta$, $\delta \in K$. Suppose that

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \exists v\, \varphi(v, x_1, \ldots, x_k). \tag{12}$$

Let $p \in g$ be such that $\{\ulcorner x_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner x_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and let Σ be a syntactical certificate for q whose associated semantical certificate is

$$\mathfrak{C}' = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle.$$

Then $\delta \in K'$ and

$$\{x_1, \ldots, x_k\} \subset X'_\delta \prec (Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}),$$

so that by (12) we may choose some $x \in X'_\delta$ with

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(x, x_1, \ldots, x_k).$$

Let $r = q \cup \{\ulcorner x \in \dot{X}_\delta \urcorner\}$.

By density, there is then some $y \in X_\delta$ such that

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(y, x_1, \ldots, x_k).$$

The proof of (C.9) is similar. Let again $\delta \in K$. Let $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta^g$ be dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta$ and

$$(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(r, x_1, \ldots, x_k). \tag{13}$$

Let $p \in g$ be such that $\{\ulcorner x_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner x_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and again let Σ be a syntactical certificate for q whose associated semantical certificate is

$$\mathfrak{C}' = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle.$$

Then $[\Sigma]^{<\omega} \cap X'_\delta$ has an element, say r, such that (13) holds true. Let $s = q \cup r \cup \{\ulcorner r \in \dot{X}_\delta \urcorner\}$.

By density, then, $g \cap X_\delta \cap E \neq \emptyset$. $\square$

Lemma 3.8 *Let g be $\mathbb{P}$-generic over V. Let*

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle$$

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let

$$N_{\omega_1} = (N_{\omega_1}; \in, A, I^*),$$

and let $T \in (\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$. Then T is stationary in $V[g]$.

If $\mathfrak{C}$, I^*, and T are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that $(\mathsf{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.8 also gives that $\mathbb{P}$ preserves the stationarity of T. In other words:

Corollary 3.9 *$\mathbb{P}$ preserves stationary subsets of ω_1.*

Proof of Lemma 3.8. Let $\dot{N}_{\omega_1} \in V^{\mathbb{P}}$ be a canonical name for N_{ω_1}, and let $\dot{I}^* \in V^{\mathbb{P}}$ be a canonical name for I^*. Let $\bar{p} \in g$, $\dot{C}$, $\dot{S} \in V^{\mathbb{P}}$, and $i < \omega_1$ and $n < \omega$ be such that

(i) $T = \dot{S}^g$,

(ii) $\bar{p} \Vdash$ "$\dot{C} \subset \omega_1$ is club,"

(iii) $\bar{p} \Vdash$ "$\dot{S} \in (\mathcal{P}(\omega_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}^*$," and

(vi) $\bar{p} \Vdash$ "$\dot{S}$ is represented by $[i, \dot{n}]$ in the term model producing $\dot{N}_{\omega_1}$."

We may and shall also assume that

$$\ulcorner \dot{N}_i \models \dot{n} \text{ is a subset of the first uncountable cardinal, yet } \dot{n} \notin \dot{I} \urcorner \in \bar{p}. \tag{14}$$

Let $p \leq \bar{p}$ be arbitrary. We aim to produce some $q \leq p$ and some $\delta < \omega_1$ such that $q \Vdash \check{\delta} \in \dot{C} \cap \dot{S}$, see Claim 3.11 below.

For $\xi < \omega_1$, let

$$D_\xi = \{q \leq p \colon \exists \eta \geq \xi \, (\eta < \omega_1 \wedge q \Vdash \check{\eta} \in \dot{C})\},$$

so that D_ξ is open dense below p. Let

$$E = \{(q, \eta) \in \mathbb{P} \times \omega_1 \colon q \Vdash \check{\eta} \in \dot{C}\}.$$

Let us write

$$\tau = ((D_\xi \colon \xi < \omega_1), E).$$

We may and shall identify τ with some subset of H_κ which codes τ.

By ($\Diamond(\mathbb{P})$), we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau). \tag{15}$$

Let h be $\mathrm{Col}(\omega, 2^{\aleph_2})$-generic over V, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and g' meets every dense set which is definable over $(N_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in N_λ. By Lemma 3.3, $\bigcup g'$ is a syntactic certificate for p, and we may let

$$\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle$$

be the associated semantic certificate. In particular, $K' \subset \lambda$.

Let S denote the subset of ω_1 which is represented by $[i, \dot{n}]$ in the term model giving N'_{ω_1}, so that if $N'_{\omega_1} = (N'_{\omega_1}, \in, A, I')$, then by (14),

$$S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'. \tag{16}$$

Let us also write $\rho = \omega_1^{V[h]} = (2^{\aleph_2})^{+V}$. Inside $V[h]$, we may extend $\langle N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle$ to a generic iteration

$$\langle N'_i, \sigma'_{ij} \colon i \leq j \leq \rho \rangle$$

such that

$$\omega_1 \in \sigma'_{\omega_1, \omega_1 + 1}(S). \tag{17}$$

This is possible as $\omega_1^{N'_{\omega_1}} = \sup\{\omega_1^{N_j} \colon j < \omega_1\} = \omega_1$ and by (16). Let

$$\langle M'_i, \pi'_{ij} \colon i \leq j \leq \rho \rangle = \sigma_{0,\rho}(\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1^{N'_0} \rangle),$$

so that $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \rho \rangle$ is an extension of $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1 \rangle$.

Recalling (7), we may lift $\langle M'_i, \pi'_{ij} \colon \omega_1 \le i \le j \le \rho\rangle$ to a generic iteration

$$\langle M^+_i, \pi^+_{ij} \colon \omega_1 \le i \le j \le \rho\rangle$$

of V. Let us write $M = M^+_\rho$ and $\pi = \pi^+_{\omega_1,\rho}$.

The key point is now that $\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \le j \le \rho\rangle$ may be used to extend $\pi\text{"}\bigcup g'$ to a syntactic certificate

$$\Sigma \supset \pi\text{"}\bigcup g' \tag{18}$$

for $\pi(p)$ in the followig manner. Let $K^* = K' \cup \{\omega_1\}$. For $\delta \in K'$, let $\lambda^*_\delta = \pi(\lambda'_\delta)$ and $X^*_\delta = \pi\text{"}X'_\delta$. Also, write $\lambda^*_{\omega_1} = \pi(\lambda)$ and $X^*_{\omega_1} = \pi\text{"}Q_\lambda$. Let

$$\mathfrak{C}^* = \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \le j \le \rho\rangle,\ \langle (k'_n, \pi(\alpha'_n)) \colon n < \omega\rangle,\ \langle \lambda^*_\delta, X^*_\delta \colon \delta \in K^*\rangle.$$

It is then straightforward to verify that $\mathfrak{C}^*$ is a semantic certificate for $\pi(p)$, and that in fact there is some syntactic certificate Σ as in (18) such that $\mathfrak{C}^*$ is certified by Σ.

Now let $[\dot m]_{\omega_1+1}$ represent $\sigma'_{\omega_1\omega_1+1}(S)$ in the term model for N'_{ω_1+1} provided by Σ, so that[13]

$$\{\ulcorner\dot\sigma_{i\omega_1+1}(\dot n) = \dot m\urcorner, \ulcorner\dot N_{\omega_1+1} \vDash \omega_1 \in \dot m\urcorner\} \subset \Sigma,$$

in other words,

$$\pi(p) \cup \{\ulcorner\dot\sigma_{i\omega_1+1}(\dot n) = \dot m\urcorner, \ulcorner\dot N_{\omega_1+1} \vDash \omega_1 \in \dot m\urcorner\} \text{ is certified by } \Sigma. \tag{19}$$

Let us now define

$$q^* = \pi(p) \cup \{\ulcorner\dot\sigma_{i\omega_1+1}(\dot n) = \dot m\urcorner, \ulcorner\dot N_{\omega_1+1} \vDash \omega_1 \in \dot m\urcorner, \ulcorner\omega_1 \mapsto \pi(\lambda)\urcorner\}. \tag{20}$$

We thus established the following.

Claim 3.10 *$q^* \in \pi(\mathbb{P})$, as being certified by Σ.*

The elementarity of $\pi \colon V \to M^+_\rho$ then gives some $\delta < \omega_1$ and some $\mu < \kappa$ such that

$$q = p \cup \{\ulcorner\dot\sigma_{i\delta+1}(\dot n) = \dot m\urcorner, \ulcorner\dot N_{\delta+1} \vDash \delta \in \dot m\urcorner, \ulcorner\delta \mapsto \lambda\urcorner\} \in \mathbb{P}. \tag{21}$$

Claim 3.11 *$q \Vdash \check\delta \in \dot C \cap \dot S$.*

[13]Here, $\dot\sigma_{i\omega_1+1}$ and $\dot N_{\omega_1+1}$ are terms of the language associated with $\pi(\mathbb{P}_\lambda)$ and $\ulcorner\dot\sigma_{i\omega_1+1}(\dot n) = \dot m\urcorner$ and $\ulcorner\dot N_{\omega_1+1} \vDash \omega_1 \in \dot m\urcorner$ are formulae of that language.

Proof of Claim 3.11. $q \Vdash \check{\delta} \in \dot{S}$ readily follows from $\{\ulcorner\dot{\sigma}_{i\delta+1}(\dot{n}) = \dot{m}\urcorner, \ulcorner\dot{N}_{\delta+1} \models \delta \in \dot{m}\urcorner\} \subset q$, the fact that $\bar{p} \geq p$ forces that $\dot{S}$ is represented by $[i, \dot{n}]$ in the term model giving $\dot{N}_{\omega_1}$, and the fact that by Claim 3.4, $[\delta]_{\delta+1}$ represents δ in the model $N_{\delta+1}$ of any semantic certificate for q.

Let us now show that $q \Vdash \check{\delta} \in \dot{C}$. We will in fact show that q forces that $\check{\delta}$ is a limit point of $\dot{C}$. Otherwise there is some $r \leq q$ and some $\eta < \delta$ such that

$$r \Vdash \dot{C} \cap \check{\delta} \subset \check{\eta}. \tag{22}$$

Let

$$\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1\rangle, \langle (k'_n, \alpha'_n) \colon n < \omega\rangle, \langle \lambda'_{\bar{\delta}}, X'_{\bar{\delta}} \colon \bar{\delta} \in K'\rangle \tag{23}$$

certify r. We must have that

(a) $\delta \in K'$,

(b) $X'_\delta \prec (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$,

(c) $X'_\delta \cap \omega_1 = \delta$, and

(d) for some Σ such that the objects from (23) are certified by Σ, $[\Sigma]^{<\omega} \cap X'_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_\lambda$ which is dense in $\mathbb{P}_\lambda \cap X'_\delta$ and definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from parameters in X'_δ.

Notice that $A_\lambda = \tau \cap Q_\lambda$, and hence A_λ may be identified with $((D_\xi \cap Q_\lambda \colon \xi < \omega_1), E \cap Q_\lambda)$. As $\eta < \delta \subset X'_\delta$, D_η is definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from a parameter in X'_δ. By (15), $D_\eta \cap Q_\lambda$ is dense in $\mathbb{P}_\lambda$. By (d) above, there is then some $s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta \cap Q_\lambda$.

By (15) again, the unique smallest $\eta' \geq \eta$ with $s \Vdash \check{\eta}' \in \dot{C}$ must be in X'_δ, hence $\eta' < \delta$ by (c) above. By Lemma 3.1, s is compatible with r. We have reached a contradiction with (22). $\square$

References

[1] D. Asperó and R. Schindler, *Bounded Martin's Maximum with an asterisk*, Notre Dame J. Formal Logic 55 (2014), pp. 333-348.

[2] B. Claverie and R. Schindler, *Increasing u_2 by a stationary set preserving forcing*, Journal of Symb. Logic 74 (2009), pp. 187-200.

[3] P. Cohen, *Set theory and the continuum hypothesis*, Benjamin, New York, 1966.

[4] S. Feferman, H. Friedman, P. Maddy, and J. Steel, *Does mathematics need new axioms?*, Bulletin of Symbolic Logic **6** (2000), pp. 401-446.

[5] Q. Feng, M. Magidor, and W.H. Woodin H., *Universally Baire Sets of Reals*, in: Judah H., Just W., Woodin H. (eds) Set Theory of the Continuum. Mathematical Sciences Research Institute Publications, vol 26, Springer, New York, NY 1992.

[6] M. Foreman, M. Magidor, and S. Shelah, *Martin's Maximum, saturated ideals and non-regular ultrafilters I*, Ann. of Mathematics 127 (1988), pp. 1 - 47.

[7] D. Hilbert, *Mathematische Probleme*, Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. Klasse (Göttinger Nachrichten), **3** (1900), pp. 253-297.

[8] R. Jensen, E. Schimmerling, R. Schindler, and J. Steel, *Stacking mice*, Journal of Symb. Logic 74 (2009), pp.315-335.

[9] P. Koellner, *The Continuum Hypothesis*, The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), E. N. Zalta (ed.), `https://plato.stanford.edu/archives/spr2019/entries/continuum-hypothesis/`

[10] P. Larson, *Martin's Maximum and the Pmax axiom* $(*)$, Annals of Pure and Applied Logic 106 (2000), pp. 135-149.

[11] P. Larson, *Martin's Maximum and definability in* $H(\omega_2)$, Annals of Pure and Applied Logic 156 (2008), pp. 110-122.

[12] M. Magidor, *Some set theories are more equal*, `http://logic.harvard.edu/EFI_Magidor.pdf`

[13] D.A. Martin and J. Steel, *A Proof of Projective Determinacy*, Journal of the American Mathematical Society 2, pp. 71-125.

[14] J. Moore, *What makes the continuum* $\aleph_2$, in: Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.), pp. 259-287.

[15] G. Sargsyan and N. Trang, *The largest Suslin Axiom*, available at `https://www.math.uci.edu/~ntrang/lsa.pdf`

[16] R. Schindler, *Woodin's axiom* $(*)$*, or Martin's Maximum, or both?*, in: Foundations of mathematics, essays in honor of W. Hugh Woodin's 60th birthday, Harvard University (Caicedo et al., eds.), pp. 177-204.

[17] J. Steel, PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$, J. Symb. Logic 70 (2005), pp. 1255-1296.

[18] J. Steel, *A stationary-tower-free proof of the derived model theorem*, in: "Advances of logic," Contemporary Mathematics 425, S. Gao, S. Jackson, and Y. Yang (eds.), Americ. Math. Society, Providence RI (2007), pp. 105-193.

[19] S. Todorcevic, *The power set of* ω_1 *and the Continuum Problem*, `http://logic.harvard.edu/Todorcevic_Structure4.pdf`

[20] W. H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter, Berlin-New York 1999.

[21] W. H. Woodin, *The Continuum Hypothesis, Part I*, Notices Amer. Math. Soc. **48 (6)** (2001), pp. 567-576.

[22] W. H. Woodin, *The Continuum Hypothesis, Part II*, Notices Amer. Math. Soc. **48 (7)** (2001), pp. 681-690.