Σ_1^1 in every real in a Σ_1^1 class or reals is Σ_1^1

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Higher Recursion Theory and Set Theory
In Honor of Theodore A. Slaman and W. Hugh Woodin
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June 3, 2019
Joint work with Ted Slaman and Leo Harrington

Setting

We work in Cantor space $2^{\mathbb{N}}$ and call its members $X \subseteq \mathbb{N}$, *reals*. We think of members of Baire space $\mathbb{N}^{\mathbb{N}}$ as functions $F : \mathbb{N} \to \mathbb{N}$ (coded as real consisting of pairs of numbers). We use the standard normal form theorems for reals and classes of reals as follows: A real X is Σ^1_1 (in a real G) if it is of the form $\{n | \exists F \forall x R(F \upharpoonright x, x, n)\}$ for a recursive (in G) predicate R. A class $\mathcal{K}$ of reals is Σ^1_1 (in G) if it is of the form $\{X | \exists F \forall x R(X \upharpoonright x, F \upharpoonright x, x)\}$ for a recursive (in G) predicate R. A real or class of reals is Δ^1_1 (or hyperarithmetic) (in G) if it and its complement are Σ^1_1 (in G).

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Main Theorem for Classes of Reals: *If a class $\mathcal{A}$ of reals is Σ^1_1 in every member G of a nonempty Σ^1_1 class $\mathcal{B}$ of reals then X is itself Σ^1_1.*

Connections to Classical Results: Basis Theorem

Gandy Basis Theorem: Every nonempty Σ^1_1 class $\mathcal{K}$ of reals contains a Z such that $\omega_1^Z = \omega_1^{CK}$.
(ω_1^Z is the least ordinal not recursive, or equivalently not Δ^1_1 in Z; ω_1^{CK} is ω_1^Z for Z recursive (or Δ^1_1).)

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Kreisel Basis Theorem: If a nonhyperarithmetic real X (i.e. X is not Δ^1_1) and $\mathcal{K} \neq \emptyset$ is Σ^1_1 then $\mathcal{K}$contains a real Z in which X is not Δ^1_1.

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Equivalently, if X is Δ^1_1 in every member of a nonempty Σ^1_1 class $\mathcal{K}$ of reals then X is Δ^1_1.

So our main theorem for reals generalizes KBT by replacing Δ^1_1 by Σ^1_1 in the second formulation. It also implies GBT: once one knows that Kleene's O is not Σ^1_1 and so there is a $Z \in \mathcal{K}$ in which O is not Σ^1_1. Spector showed that this implies that $\omega_1^Z = \omega_1^{CK}$.

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Upon hearing about our results Simpson and Steel each informed us of an analogous theorem at a different level of the hierarchies.

Analogs

Theorem (Andrews and J. Miller): Let P be a nonempty Π^0_1 class. If X is Π^0_1 in every member of P then X is Π^0_1. (Or, equivalently, if X is Σ^0_1 in every member of P then X is Σ^0_1.)

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At the Σ^1_2 level note a classical basis theorem: Every nonempty Σ^1_2 class of reals contains a Δ^1_2 real. Of course, any real Σ^1_2 in a Δ^1_2 real is itself Σ^1_2. So we have the analog for our main theorem with Σ^1_2 replacing Σ^1_1.

Hyperarithmetic Theory Applications

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More generally, can be phrased in terms of ω-models:

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In all of these results it is easy to see that the class of models described is Σ^1_1 and, of course, every member X of such a model is recursive in it and so any real in every such model is Σ^1_1 but these models are all trivially closed under complementation. So these Theorems all follow from our Main Theorem for reals.

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The type is *realized* in a structure $\mathcal{M}$ for the language if there are elements $a_1, \ldots, a_n$ of the structure such that $p = \{\varphi \in \Gamma | \varphi$ has free variables $x_1, \ldots, x_n$ & $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)\}$.

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If p is not realized in $\mathcal{M}$ we say it is *omitted* in $\mathcal{M}$. (Note that this definition is more general than the usual definition of an n-type for the language.)

$\mathcal{N}$ -Logics and Omega Models

We begin with a class of logics somewhat more general than ω-logic. We consider two sorted logics $(\mathcal{N}, \mathcal{M}, \ldots)$ in the usual sense of having two types of variables one ranging over the elements of $\mathcal{N}$ and the other over those of $\mathcal{M}$ in addition to the usual apparatus of function, relation and constant symbols of ordinary first order logic. While formally merely a version of first order logic gotten by adding on predicates for N and M, this logic can be turned into a much stronger one ($\mathcal{N}$-logic) by requiring that all models have their first sort (with some functions and relations on it as given in the structure) isomorphic to some given countable first order structure.

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The most common example of these logics is ω-logic where we require that $\mathcal{N}$ be isomorphic to the ordinal ω or the standard model $\mathbb{N}$ of arithmetic (depending on the language intended). Again, the most common examples are given by classes of ω-models of fragments T of second order arithmetic.

Type Omitting for $\mathcal{N}$ -Logic

As being an $\mathcal{N}$-model, or even one also satisfying some Π^1_1 theory T, is clearly Σ^1_1 in $\mathcal{N}$, we immediately get all the results mentioned above and more as corollaries of our theorem.

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Theorem: If T is a Π^1_1 set of sentences of $(\mathcal{N}, \mathcal{M}, \ldots)$; $\mathcal{N}$ is a countable structure for the appropriate sublanguage (for the first sort); T has an $\mathcal{N}$-model; Γ is a Σ^1_1 in $\mathcal{N}$ set of formulas of the language of $(\mathcal{N}, \mathcal{M}, \ldots)$ (with free variables $x_1, \ldots, x_n$) and p is a Γ- n-type which is not Σ^1_1 in $\mathcal{N}$, then there is an $\mathcal{N}$-model of T not realizing p.

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Proof: By a Skolem-Löwenheim type argument, if T has an $\mathcal{N}$-model it has a countable $\mathcal{N}$-model. Being a countable $\mathcal{N}$-model of T is Σ^1_1 in $\mathcal{N}$ and so by our Theorem (relativized to $\mathcal{N}$) there is an $\mathcal{N}$-model $(\mathcal{N}, \mathcal{M}, \ldots)$ of T in which p is not even Σ^1_1 in $\mathcal{N}$. Of course, any Γ-n-type realized in $(\mathcal{N}, \mathcal{M}, \ldots)$ is hyperarithmetic in $(\mathcal{N}, \mathcal{M}, \ldots)$.

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Of course, we can relativize this theorem as well to any real C. As a sample application we build ω-models of ZFC controlling the well-founded part.

Omega Models of ZFC

Corollary: For every real C and reals X_n not Δ^1_1 in C, there is a countable ω-model of ZFC containing C but not containing any X_n whose well founded part consists of the ordinals less than ω_1^C, the first ordinal not recursive in C.

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Now apply the Theorem on omitting sequences of types first adding on a new real $X_0 = O^C$ (i.e. Kleene's O relativized to C) to the list.

It supplies a countable ω-model of ZFC containing C but not containing any of the X_n. As it contains C it contains every ordering recursive in C and so order types for every ordinal less than ω_1^C. On the other hand, if there were an ordinal in the model isomorphic to ω_1^C then, by standard results of hyperarithmetic theory, O^C would be in the model as well.

Some Other Logics Between First and Second Order

Type omitting theorems (some known others perhaps not) for several related logics between first and second order are consequences of our theorem in the same way.

Weak second order logic is second order logic where the second order quantifiers range over finite subsets of the domain.

Cardinality logic (for $\aleph_0$) adds a new quantifier Q_0 to first order logic and interprets $Q_0 x \varphi(x)$ to mean that there are infinitely many x such that $\varphi(x)$ holds.

Ancestral logic adds the transitive closure operation to first order logic by introducing a new operator (quantifier) TC, extending the syntax by making $TC_{x,y}\varphi(x,y)(u,v)$ a formula with new free variables u and v for every ordinary formula φ and variables x, y (which become bound in this formula). The semantics are determined by saying that $TC_{x,y}\varphi(x,y,u,v)(a,b)$ holds if there is a sequence of elements $a = c_0, \ldots c_n = b$ such that $\varphi(c_i, c_{i+1})$ holds for every $i < n$.

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Corollary: Consider the languages of any of weak second order logic, cardinality logic (for $\aleph_0$), ancestral logic or $\mathcal{N}$-logic (for a countable $\mathcal{N}$). If T and Γ are, respectively, Π^1_1 and Σ^1_1 (in $\mathcal{N}$) sets of sentences in the appropriate language, T has a model (for the appropriate semantics), and $\{p_i\}$ is a set of Γ- n_i-types none of which is Σ^1_1 (in $\mathcal{N}$) then there is an ($\mathcal{N}$-)model of T not realizing any of the p_i.

Computable Infinitary Logic

A similar argument works for computable infinitary logic $\mathcal{L}_c$ based on a (wlog recursive) first order language $\mathcal{L}$ if one takes care of the issue that the infinitary languages are no longer recursive (or even hyperarithmetical which would work as well). We omit the detailed definition of this logic.

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Thus if T is a Π^1_1 set of sentences of $\mathcal{L}_c$, a structure $\mathcal{M}$ (for $\mathcal{L}$) then being a model of T is a Σ^1_1 property of $\mathcal{M}$.

A subset Γ of $\mathcal{L}_c$ is Σ^1_1 on $\mathcal{L}_c$ if $\Gamma = \mathcal{L}_c \cap S$ for some $S \in \Sigma^1_1$ or equivalently if $\Gamma \cup \{n | n \notin \mathcal{L}_c\}$ is Σ^1_1. For example, the set Γ_α of formulas of $\mathcal{L}_c$ of level at most α is Σ^1_1 on $\mathcal{L}_c$ (in fact it is a Δ^1_1 set). As before we have the notion of p being a Γ-n-type. We thus immediately have the appropriate type omitting theorems for computable infinitary languages.

Omitting Types for $\mathcal{L}_c$

Corollary: If T is a Π^1_1 set of sentences of $\mathcal{L}_c$, T has a model (and so a countable model), Γ is is Σ^1_1 on $\mathcal{L}_c$ and $\{p_i\}$ is a set of Γ-n_i-types of $\mathcal{L}_c$ none of which is Σ^1_1 on $\mathcal{L}_c$ then there is a model of T not realizing any of the p_i. For example, if $\{p_i\}$ is a set of Γ_{α_i}- n_i-types of $\mathcal{L}_c$ none of which is Σ^1_1 then there is a model of T not realizing any of the p_i.

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Proof: Let S witness that Γ is Σ^1_1 on $\mathcal{L}_c$, i.e. S is Σ^1_1 and $\Gamma = \mathcal{L}_c \cap S$. If p_i were realized in $\mathcal{M}$ then there would be $a_1, \dots a_{n_i}$ in $\mathcal{M}$ such that $p_i = \{\varphi(x_1, \dots x_{n_i}) \in \mathcal{L}_c \cap S | \mathcal{M} \vDash \varphi(a_1, \dots a_{n_i})\} = \mathcal{L}_c \cap S \cap \{n | n \notin \mathcal{L}_c \vee n$ is the code for a formula $\varphi(x_1, \dots x_{n_i})$ and $\mathcal{M} \vDash \varphi(a_1, \dots a_{n_i})\}$. As $\mathcal{L}_c$ is Π^1_1 and the required manipulations on formulas are hyperarithmetic and satisfaction in $\mathcal{M}$ is Σ^1_1, p_i would be Σ^1_1 on $\mathcal{L}_c$ contrary to our assumption.

Modal Logics

We next consider modal logics such as $\mathcal{L}^{\Box,\Diamond}$ with semantics given by Kripke frames $\mathcal{F} = (W, S, \mathcal{C}(p))$ consisting of a set W of worlds p, an accessibility relation S on W and a collection $\{\mathcal{C}(p) | p \in W\}$ of classical structures for a first order language $\mathcal{L}$.

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Here we can derive type omitting theorems that allow the class of classical models considered to be specified by a set of sentences of $\mathcal{L}^{\Box,\Diamond}$ but also by specifications on the whole frame that allow us to say, for example, that a modal sentence is forced in some world, every world or some collection of world with some characterization.

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Similarly, we can impose requirements on the accessibility relation in the allowed frames. Some standard such restrictions on the accessibility relation can also be captured by sentences of $\mathcal{L}^{\Box,\Diamond}$ but others are more complicated. For example, frames in which the accessibility relation on worlds is precisely $<$ on $\mathbb{N}$ or some other fixed countable relation or some class of relations with another characterization.

Omitting Types for Modal Logics

One approach to types here is to consider a Γ as above contained in $\mathcal{L}^{\Box,\Diamond}$ and to say that a Γ-n-type q is realized in a frame $\mathcal{F}$ if there is a $p \in W$ and and $a_1, \ldots, a_n \in \mathcal{C}(p)$ such that $p = \{\varphi \in \Gamma | \varphi$ has free variables $x_1, \ldots, x_n$ & $p \Vdash \varphi(a_1, \ldots, a_n)\}$.

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Corollary: Let T be a Π^1_1 set consisting of Σ^1_1 sentences about the accessibility relation S and Σ^1_1 sentences about the relation $p \Vdash \varphi$ where p ranges over W and φ ranges over sentences of $\mathcal{L}^{\Box,\Diamond}$ such that there is is a countable frame making all of these sentences true. If $\{q_i\}$ is a set of Γ-n_i-types none of which is Σ^1_1, then there is a frame in which all the sentences of T are true not realizing any of the q_i.

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We can say more by allowing types to restrict the accessibility relation and the whole frame. We can also allow the language and structure at each world to be appropriate for one of the generalized logics above.

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We can say more by allowing types to restrict the accessibility relation and the whole frame. We can also allow the language and structure at each world to be appropriate for one of the generalized logics above.
Question: Has anyone studied modal logics where the structures at each node are one for fragments of second order logic as above?

Extension Logics

We suggest a logic that captures the notions of being able to extend a structure by adding on new relation symbols satisfying given axioms.

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Corollary: If T is a Π^1_1 set of formulas of $\mathcal{L}^{Ext}$ with a model, Γ a Σ^1_1 set of formulas of $\mathcal{L}^{Ext}$ as before and $\{p_i\}$ is a set of n_i-types of $\mathcal{L}^{Ext}$ none of which is Σ^1_1 then there is a model $\mathcal{M}$ of T in which no p_i is realized.

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Basically, this says that, with our usual restrictions on the $\mathcal{L}$-theories T, that there is a model which can be extended by relations satisfying additional axioms involving the new relations but cannot be further extended to one satisfying any one of a collection of sentences p_i in $\mathcal{L}^{Ext}$

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Question: Has anyone seen a logic like this? Do any "practical" instances or applications come to mind?

Complexity of the Reals and Models

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Theorem: If $\mathcal{K}$ is a nonempty Σ^1_1 class reals and X_n a countable sequence of reals uniformly Δ^1_1 (recursive) in O none of which is Σ^1_1, then there is a $G \in \mathcal{K}$ with G Δ^1_1 (recursive) in O such that no X_n is Σ^1_1 in G. Indeed, G can be chosen to be of strictly smaller hyperdegree than O, i.e. O is not Δ^1_1 in G. As in Theorem for sequences of reals, if we assume only that the X_n are not Δ^1_1 then we may conclude that none is Δ^1_1 in G.

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As a sample application we give a common generalization combining both KBT and GBT.

Kleene and Gandy Basis Theorems

Note that by a result of Spector's, $\omega_1^{CK} < \omega_1^A$ implies that O is Δ_1^1 in A (indeed there is a pair of Σ_1^1 formulas $\varphi(X, n)$ and $\theta(X, n)$ which define O and its complement for any X with $\omega_1^X > \omega_1^{CK}$), we thus simultaneously have the Kleene and Gandy basis theorem for Σ_1^1 classes as well.

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Theorem: Every nonempty Σ_1^1 class of reals $\mathcal{K}$ contains an element A recursive in and of strictly smaller hyperdegree than O. In particular, one with $\omega_1^A = \omega_1^{CK}$.

Classes of Reals

Theorem: If a class $\mathcal{A}$ of reals is Σ^1_1 in every member of a nonempty Σ^1_1 class $\mathcal{B}$ of reals then it is Σ^1_1.

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Corollary: Any class $\mathcal{A}$ of reals which is Δ^1_1 in every member of a Σ^1_1 class $\mathcal{B}$ of reals is Δ^1_1.
Question: Does the last theorem hold with Σ^1_1 replaced by Δ^1_1?
Question: Are there any descriptive set theory applications of these results?

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We view the Σ^1_1 formulas $\varphi(G, n)$ as of the form $\exists F \forall x R(G \upharpoonright x, F \upharpoonright x, x, n)$ with R recursive. We say that $\mathcal{L} \Vdash \varphi(G, n)$ if $(\forall Z \in \mathcal{L})(\varphi(Z, n))$. If, as usual, we say $\mathcal{L} \Vdash \neg\varphi(G, n)$ if $(\forall \hat{\mathcal{L}} \subseteq \mathcal{L})(\hat{\mathcal{L}} \nVdash \varphi(G, n))$, this definition is then equivalent to $(\forall Z \in \mathcal{L})(\neg\varphi(Z, n))$. The point here is that if there is a $Z \in \mathcal{L}$ such that $\varphi(Z, n)$ then $\hat{\mathcal{L}} = \mathcal{L} \cap \{Z | \varphi(Z, n)\}$ is a nonempty extension of $\mathcal{L}$ which obviously forces $\varphi(G, n)$.

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We now list all the Σ_1^1 formulas $\Theta_k(G, n)$. These are the formulas that could potentially define the reals Σ_1^1 in any G.

Plan of Proof

We consider an X which is a candidate for being Σ^1_1 in every $G \in \mathcal{K}$. We build a sequence $\mathcal{L}_k$ of conditions beginning with $\mathcal{L}_0 = \mathcal{K} = \{G | \exists F_0 \forall x R_{m_0}(G \upharpoonright x, F_0 \upharpoonright x, x)\}$ as well as initial segments γ_k (of length at least k) of our intended G and $\delta_{i,k}$ of witnesses F_i (of length at least k) showing that $G \in \mathcal{L}_k$. More precisely, each $\mathcal{L}_k$ will be of the form $G \supset \gamma_k \ \& \ \forall i \leq k \exists F_i \supset \delta_{i,k} \forall x R_{m_i}(G \upharpoonright x, F_i \upharpoonright x, x)$ for some recursive R_{m_i} (independent of k).

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Thus, if we successfully continue our construction keeping $\mathcal{L}_k$ nonempty for each k then the $F_i = \lim_k \delta_{i,k}$ for $i \leq k$ will witness that $G = \lim_k \gamma_k$ is in every $\mathcal{L}_k$ as we guarantee that $R_{m_i}(\gamma_k \upharpoonright x, \delta_{i,k} \upharpoonright x, x)$ holds for every $i, x < k$ and every k.

The Construction

We begin with $\gamma_0 = \emptyset = \delta_{0,0}$ and R_{m_0} as specified by $\mathcal{K}$. So our G will at least be in $\mathcal{K}$ as desired. Suppose we have defined γ_j and $\delta_{i,j}$ for $j, i \leq k$ and wish to define $\mathcal{L}_{k+1}$, γ_{k+1} and $\delta_{i,k+1}$ for $i \leq k+1$ so as to prevent X from being Σ^1_1 in G via Θ_k. We ask if there is an $m \in \omega$ and a nonempty $\mathcal{L} \subseteq \mathcal{L}_k$ such that

1. $m \notin X$ and $\mathcal{L} \Vdash \Theta_k(G, m)$ or
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Suppose there is such an $\mathcal{L}$ of the form $\exists F_{k+1} \forall x R_{m_{k+1}}(G \restriction x, F_{k+1} \restriction x, x)$. As $\mathcal{L} \subseteq \mathcal{L}_k$ is nonempty we can choose $\gamma_{k+1} \supset \gamma_k$ and $\delta_{i.k+1} \supset \delta_{i,k}$ for $i \leq k$ and some $\delta_{k+1,k+1}$ all of length at least $k+1$ such that $\mathcal{L}_{k+1}$ as given by $G \supset \gamma_{k+1}$ & $(\forall i \leq k+1)(\exists F_i \supset \delta_{i,k+1})(\forall x R_{m_i})(G \restriction x, F_i \restriction x, x)$ is a nonempty subclass of $\mathcal{L}$ (and so, in particular, $R_{m_i}(\gamma_{k+1} \restriction x, \delta_{i,k+1} \restriction x, x)$ for every $i, x \leq k+1$). We can now continue our induction.

First Outcome: All successes

Note that if we can successfully define nonempty $\mathcal{L}_k$ in this way for every k then we build a $G = \lim_k \gamma_k$ and $F_i = \lim_k \delta_{i,k}$ for each i such that $\forall x R_{m_i}(G \restriction x, F_i \restriction x, x)$. In particular $\forall x R_{m_0}(G \restriction x, F_0 \restriction x, x)$ and so $G \in \mathcal{K}$. Similarly, $G \in \mathcal{L}_k$ for every $k > 0$. If X is $\Sigma^1_1(G)$ as assumed, then $X = \{n | \Theta_k(G, n)\}$ for some k. We consider the construction at stage $k+1$ and the $\mathcal{L}$ chosen at that stage. If we were in case (1) then as $\mathcal{L} \Vdash \Theta_k(G, m)$ and $G \in \mathcal{L}_{k+1}$, $\Theta(G, m)$ is true but $m \notin X$ for a contradiction. Similarly, if we were in case (2), as $\mathcal{L} \Vdash \neg\Theta_k(G, m)$ and $G \in \mathcal{L}_{k+1}$, $\neg\Theta(G, m)$ is true but $m \in X$ again for a contraction.

Second Outcome: A First Failure

Thus we can assume that there is some first stage $k+1$ at which there are no m and $\mathcal{L} \subseteq \mathcal{L}_k$ as required in the construction. In this case we claim that X is Σ^1_1 as desired. Indeed, we claim that X is defined as a Σ^1_1 real by $m \in X \Leftrightarrow (\exists Z \in \mathcal{L}_k)\Theta_k(Z, m)$. To see this suppose first that $(\exists Z \in \mathcal{L}_k)\Theta_k(Z, m)$. Then $\mathcal{L}$ as defined by $\mathcal{L}_k$ & $\Theta_k(G, m)$ is a nonempty Σ^1_1 class such that $\mathcal{L} \Vdash \Theta_k(G, m)$ and so we would have $m \in X$ as desired by the assumed failure of (1) at stage $k+1$ of the construction. On the other hand, if $(\forall Z \in \mathcal{L}_k)(\neg\Theta_k(Z, m)$ then $\mathcal{L}_k \Vdash \neg\Theta_k(G, m)$ and so by the failure of (2) at stage $k+1$ of the construction, $m \notin X$ as desired.

Proof for Sequences Version

Repeat the proof of the Main Theorem but at step $k+1 = \langle n, j \rangle$ of the construction replace X by X_n and Θ_k by Θ_j. If we successfully pass through all steps k then the previous argument shows that no X_n is Σ_1^1 in $G \in \mathcal{K}$. On the other hand, if the construction terminates at step $k+1 = \langle n, j \rangle$ then the previous argument shows that X_n is defined as a Σ_1^1 real by $m \in X_n \Leftrightarrow (\exists Z \in \mathcal{L}_k)\Theta_j(Z, m)$ for a contradiction. For the Δ_1^1 version, simply consider the sequence Y_n where $Y_n = X_n$ if X_n is not Σ_1^1 and Y_n is the complement of X_n otherwise (i.e. X_n is not Π_1^1). As now no Y_n is $\Sigma_1^1(G)$, no X_n is $\Delta_1^1(G)$.

Complexity Calculations

Suppose we are at step $k = \langle n, j \rangle$ of the construction. We know that either there is an $m \in X_n$ such that $(\forall Z \in \mathcal{L}_k)(\neg\Theta_k(Z, m))$ or an $m \notin X_n$ such that $(\exists Z \in \mathcal{L}_k)(\Theta_k(Z, m))$. As the X_n are uniformly Δ^1_1 (recursive) in O, and the rest of the conditions considered in the construction are either Σ^1_1 or Π^1_1, O can hyperarithmetically (recursively) decide which case to apply. As choosing the $\gamma_{k+1} \supset \gamma_k$ and $\delta_{i.k+1} \supset \delta_{i,k}$ for $i \leq k$ and so $\mathcal{L}_{k+1}$ now only require finding ones for which the corresponding Σ^1_1 class $\mathcal{L}_{k+1}$ is nonempty, this step is also recursive in O. Of course, as we can add O onto the list of X_n, we then guarantee that O is not Σ^1_1 in G and so, of course, not Δ^1_1 in G as required.