Finding Randomness

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Introduction

1. Finding Randomness
   1.1 Lebesgue measure
      1.1.1 Formulated by measure
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Lebesgue Measure

formulated by measure

Definition

A real number ξ is *Martin-Löf random* if it does not belong to any effectively-null G_δ set. Precisely, if $(O_n : n \in \mathbb{N})$ is a uniformly computably enumerable sequence of open sets such that for all n, O_n has measure less than $1/2^n$, then $\xi \notin \cap_{n\in\mathbb{N}} O_n$.

This is not mysterious: Identify a family of sets of measure 0, and say that ξ is random if it does not belong to any set in the family.

Randomness

formulated by compressibility

Definition

A real number ξ is *algorithmically incompressible* iff there is a C such that for all ℓ, $K(\xi \restriction \ell) > \ell - C$, where K denotes prefix-free Kolmogorov complexity and $\xi \restriction \ell$ denotes the first ℓ bits in the base 2 representation of ξ.

This is also not mysterious: Say that ξ is incompressible when for all ℓ, it takes ℓ bits of information to describe $\xi \restriction \ell$. One can interpret *description* in a variety of ways and obtain a reasonable characteristic of ξ.

Schnorr's Theorem

Theorem (Schnorr 1973)

ξ is Martin-Löf random iff it is algorithmically incompressible.

Representing Measures other than Lebesgue Measure

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$X \in 2^\omega$ is n-random relative to a representation m of μ if and only if it does not belong to any $m^{(n-1)}$-presented G_δ set of μ-measure 0.

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We will drop the explicit reference to presentations and speak of randomness relative to μ.

Arbitrary Measures and 1-Randomness

Theorem

For $X \in 2^\omega$, the following are equivalent.

- *X is not recursive.*
- *There is a measure μ such that $\mu(\{X\}) = 0$ and X is 1-random relative to μ.*

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Comments on the construction of μ, given X not recursive:

- Apply the Posner-Robinson Theorem to find G such that $X + G \equiv G'$.
- Note that $G' \equiv_T R$, where R is 1-random relative to G.
- By compactness, convert the Turing equivalence between X and R into a push-forward of Lebesgue measure to another measure μ so that R's 1-randomness transforms into X's 1-randomness for μ.

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Theorem

For all $n \geq 1$, for all but countably many $X \in 2^\omega$, there is a continuous measure μ such that X is n-random relative to μ.

Theorem

For all k, the previous theorem cannot be proven in $ZF^- + k$-many iterates of the power set of ω.

Continuous Measures

degree theoretically characterizing relative randomness

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- For X, Y, and Z in 2^ω, we write $X \equiv_{T,Z} Y$ to indicate that there are Turing reductions (i.e. representations of continuous functions) Φ and Ψ which are recursive in Z such that $\Phi(X) = Y$ and $\Psi(Y) = X$.
- When Φ and Ψ have domain 2^ω, we write $X \equiv_{tt,Z} Y$.

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Turing reductions correspond to continuous functions defined on subsets of 2^ω. Truth-table (tt) reductions correspond to continuous functions defined on all of 2^ω.

Continuous Measures

degree theoretically characterizing relative randomness

Proposition

For X and Z in 2^ω, the following conditions are equivalent.

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- *There is an R such that R is n-random relative to Z and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that f is recursive in Z and $f(R) = X$.*

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- *There is an R such that R is n-random relative to Z and $X \equiv_{tt,Z} R$.*

Constructing continuous measures

In order to conclude that X is n-random relative to some continuous measure, it is sufficient to find a Z relative to which X is tt-equivalent to some n-random sequence R.

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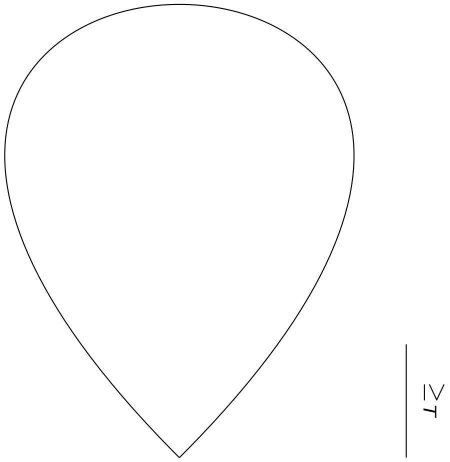
Example

If X is recursive, then X is not 1-random relative to any continuous measure.

2^ω ordered by $\geq_T$



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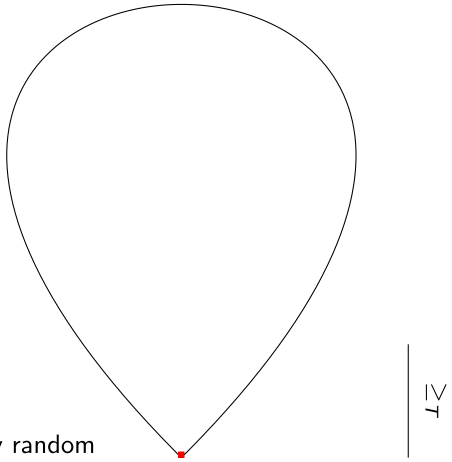

■ not relatively random

Constructing continuous measures

Theorem (Martin, Borel Determinacy)

Suppose that $\mathcal{B}$ is a Borel subset of 2^ω and that for every A there is a Y such that $Y \geq_T A$ and $Y \in \mathcal{B}$. There is a $B \in 2^\omega$ such that for every $X \geq_T B$ there is a Y such that $Y \equiv_T X$ and $Y \in \mathcal{B}$.

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Corollary

For any $n \geq 1$, there is a B such that for all $X \geq_T B$, there is a continuous measure μ such that X is n-random relative to μ.

- If X is Turing equivalent to an $(n+1)$-random relative to Z then X is *tt*-equivalent to an n-random relative to Z'.
- Consider the set $\mathcal{B}$ of Y's of the form $A + R$, where R is $(n+1)$-random relative to A. These are all n-random relative to some continuous measure.

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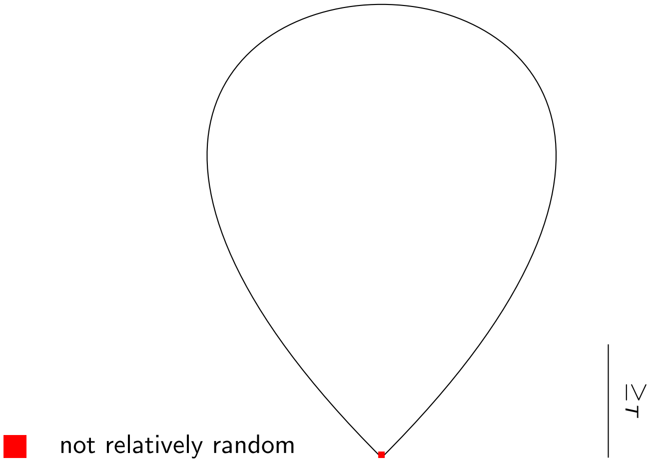

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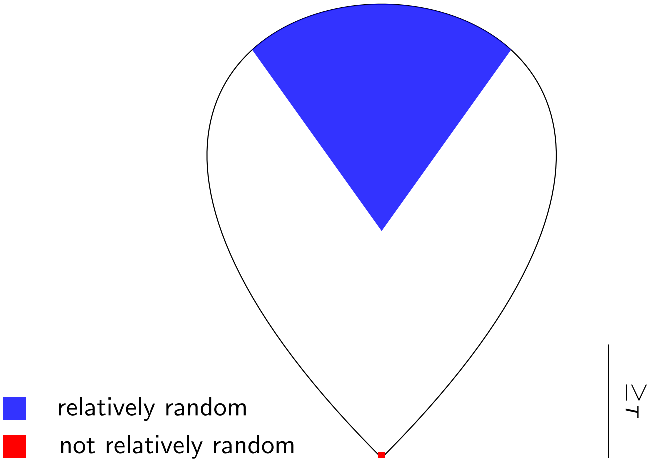

Constructing continuous measures

Martin's proof implies that if G is a real parameter used to define a cofinal Borel set $\mathcal{B}$, then the B for that set belongs to the smallest countable model of a sufficiently large subset of ZFC, the axioms of set theory, to which G belongs.

Constructing continuous measures

Fix n and let $L_{\lambda(n)}$ be the smallest countable model satisfying ZFC^-, set theory without the power set axiom, and the existence of n-iterates of the power set applied to $\mathbb{R}$.

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Theorem

Suppose that $X \notin L_{\lambda(n)}$. Then there is a G such that

- *$L_{\lambda(n)}[G]$ is a model of ZFC^- and the existence of n-iterates of the power set applied to $\mathbb{R}$.*
- *Every element of $2^\omega \cap L_{\lambda(n)}[G]$ is recursive in $X + G$.*

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- Proof by Kumabe-Slaman forcing.
- Consequently, if $X \notin L_{\lambda(n)}$, then relative to G, X is in the cone of relatively random reals.

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Theorem (Co-countability)

For all n, for all but countably many $X \in 2^\omega$ there is a continuous measure μ such that X is n-random relative to μ.

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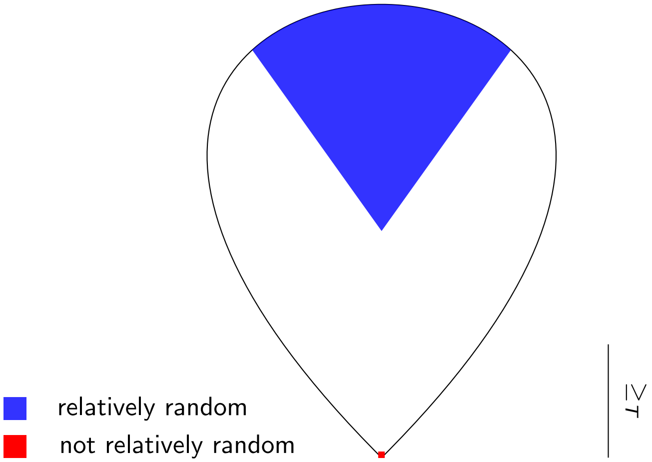

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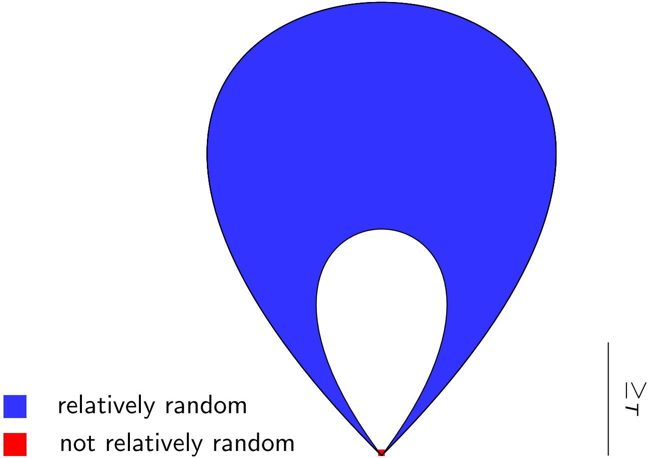

The Empty Bubble and the Necessity of Power Sets

- We will exhibit lower bounds on the scope of the empty bubble of the previous slide.
- It will follow that infinitely many iterates of the power set of ω are needed to prove the co-countability theorem above.
  - The proof sketch above invoked Turing determinacy for arithmetic subset of 2^ω, which is well-known by work of H. Friedman to have this property.

Within the Bubble

a little more about random sequences

Suppose that $n \geq 2$, $Y \in 2^\omega$, and X is n-random relative to μ.

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Suppose that $n \geq 2$, $Y \in 2^{\omega}$, and X is n-random relative to μ.

If i is less than n, Y is recursive in $(X + \mu)$ and recursive in $\mu^{(i)}$, then Y is recursive in μ.

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In general, using arithmetic definitions with fewer than n quantifiers, n-random reals do not accelerate arithmetic definability.

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Example

For all k, $0^{(k)}$ is not 2-random relative to any μ.

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- Say $0^{(k)}$ is 2-random relative to μ.

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- Say $0^{(k)}$ is 2-random relative to μ.
- $0'$ is recursively enumerable relative to μ and recursive in the supposedly 2-random $0^{(k)}$. Hence, $0'$ is recursive in μ and so $0''$ is recursively enumerable relative to μ.

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- Use induction to conclude $0^{(k)}$ is recursive in μ, a contradiction.

□

Within the Bubble

a little more about random sequences

Definition

A linear order $\prec$ on ω is *well-founded* iff every non-empty subset of ω has a least element.

As with arithmetic definability, for $n \geq 5$, n-random reals cannot accelerate the calculation of well-foundedness.

Within the Bubble

a little more about set theory

Definition

Gödel's hierarchy of constructible sets L is defined by the following recursion.

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathit{Def}(L_\alpha)$, the set of subsets of L_α which are first order definable in parameters over L_α.
- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.

Within the Bubble

a little more about set theory

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  - M_β is obtained from smaller M_α's by iterating the Turing jump and taking arithmetically definable direct limits.
  - Every $X \in 2^\omega \cap L_\lambda$ is recursive in some M_β.

Master Codes and Effective Randomness

failures of continuous randomness

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Suppose that M_β were random relative to μ.

1. Consider the structures recursively presented relative to μ which satisfy $V = L$ and their master codes.
2. Extract a maximal well-ordered set of those which appear well-founded, which will be an initial segment of the master codes below β.
3. Use a generalization of the jump argument for a contradiction.

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Corollary

L_λ does not satisfy the Co-countability Theorem. Hence, ZFC^- does not prove the Co-countability Theorem.

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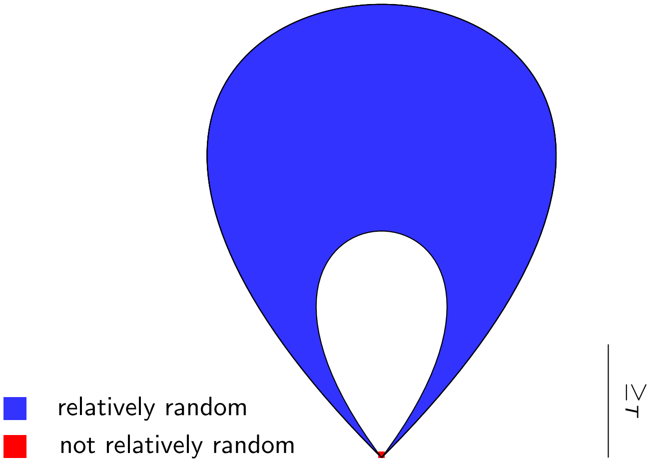

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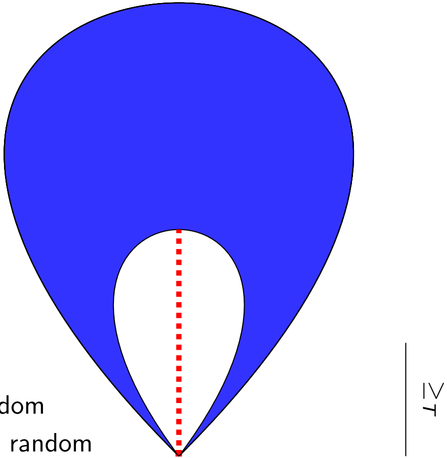

relatively random

not relatively random

A Manifesto

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- In Recursion Theory, the universal sets for the iteration of the existential number quantifier, i.e. the master codes, embody self-generating structure. By the above, they have an identifiable lack of randomness.
- Further, these same sets generate all failures of relative randomness.

Questions

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4. **Due to Andrew Marks and Adam Day.** Say that a measure is *awesome* if it is the push-forward of Lebesgue measure by a continuous Turing-invariant injection from 2^ω to 2^ω. Say that X and Y are n-related if they are both n-random relative to the same awesome measure. Let R_n be the transitive closure of this relation.
   4.1 Is there an X which is R_n-equivalent to a set of Turing degree X'?
   4.2 Is there an R_n equivalence class that is cofinal in the Turing degrees?