Finding Better Randomness

Theodore A. Slaman

University of California Berkeley

Outline

1. Measures with the mass distribution property and Hausdorff dimension
2. Measures with well-behaved Fourier transforms and Fourier dimension
3. Diophantine Approximation
    3.1 Randomness formulated by normality
    3.2 Randomness formulated by irrationality exponent

Hausdorff Dimension (Size for Null Sets)

Hausdorff dimension is usually defined in terms of open covers, but the following is equivalent by Frostman's Lemma (1935).

Definition

- For $s \in [0, 1]$ and A a Borel set of real numbers, A has Hausdorff dimension at least s iff there is a Borel measure μ and a positive constant C such that $\mu(A) > 0$ and for all reals ξ and $r > 0$, $\mu(B(\xi, r)) \leq C \cdot r^s$.
- The Hausdorff dimension of A is the supremum of such s.

Hausdorff Dimension (Size for Null Sets)

Hausdorff dimension is usually defined in terms of open covers, but the following is equivalent by Frostman's Lemma (1935).

Definition

- For $s \in [0, 1]$ and A a Borel set of real numbers, A has Hausdorff dimension at least s iff there is a Borel measure μ and a positive constant C such that $\mu(A) > 0$ and for all reals ξ and $r > 0$, $\mu(B(\xi, r)) \leq C \cdot r^s$.
- The Hausdorff dimension of A is the supremum of such s.

When the first condition holds, we say that μ is *s-regular* or that μ has the *Mass Distribution Property for s.*

Hausdorff Dimension (Size for Null Sets)

Example

- The Cantor middle-third set has Hausdorff dimension $\log(2)/\log(3)$.
- Its uniform measure is $\log(2)/\log(3)$-regular.

Effective Hausdorff Dimension

Effective Hausdorff dimension is usually defined for a subset of $\mathbb{R}$ in terms of martingales (Lutz 2000) or effectively presented open covers (Reimann 2004), but the following is equivalent for singletons $\{\xi\}$ by a theorem of Mayordomo (2017).

Definition (Lutz, Mayordomo)

The *effective Hausdorff dimension* of a real number ξ is the infimum of the numbers r such that for infinitely many ℓ the sequence of the first ℓ digits in the binary expansion of ξ has prefix-free Kolmogorov complexity less than or equal to $r \cdot \ell$.

Effective Hausdorff Dimension

Remark

- $\{\xi : \xi$ *has effective Hausdorff dimension* $d\}$ *has Hausdorff dimension* d.

Effective Hausdorff Dimension

Remark

- *$\{\xi : \xi$ has effective Hausdorff dimension $d\}$ has Hausdorff dimension d.*
- *(J. Lutz and N. Lutz 2017) For $A \subseteq \mathbb{R}$, the Hausdorff dimension of A is equal to*
  the infimum over all $B \subseteq \mathbb{N}$
  of the supremum over all $\xi \in A$
  of the effective-relative-to-B Hausdorff dimension of ξ.

Randomness for s-Regular Measures

Theorem (Reimann (2008))

Suppose that $\xi \in [0,1]$ has effective dimension d. For all $s < d$, there is an s-regular measure μ such that ξ is 1-random relative to μ.

Fourrier Dimension

The Fourier-Stieltjes transform

Definition

The Fourier transform $\widehat{\mu}$ of a finite Borel measure μ on $\mathbb{R}$ is given by:

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{-2\pi i t \cdot x} \, d\mu(x).$$

When $d\mu = f(x)dx$, this is the same as the Fourier transform of f.

Fourier Dimension (Uniform Distribution of Measure)

Definition

The *Fourier dimension* of a set $A \subseteq \mathbb{R}$ is is the supremum of the $s \leq 1$ such that there is a measure μ with support A and a positive constant C such that for all $t \in \mathbb{R}$, $|\widehat{\mu}(t)| \leq C \cdot (1 + |t|)^{-s/2}$.

Fourier Dimension (Uniform Distribution of Measure)

Definition

The *Fourier dimension* of a set $A \subseteq \mathbb{R}$ is is the supremum of the $s \leq 1$ such that there is a measure μ with support A and a positive constant C such that for all $t \in \mathbb{R}$, $|\widehat{\mu}(t)| \leq C \cdot (1 + |t|)^{-s/2}$.

- For today, we shall say that μ as above is a Fourier measure for dimension s.

Fourier Dimension (Uniform Distribution of Measure)

- The Fourier dimension of a set is less than or equal to its Hausdorff dimension.
  - The Cantor middle-third set has Fourier dimension zero.

Fourier Dimension (Uniform Distribution of Measure)

- The Fourier dimension of a set is less than or equal to its Hausdorff dimension.
  - The Cantor middle-third set has Fourier dimension zero.
- $A \subseteq \mathbb{R}$ is a *Salem* set iff its Fourier dimension is equal to its Hausdorff dimension.
  - $\{\xi : \xi$ has effective Hausdorff dimension $d\}$ is a Salem set.
  - Fourier dimension is difficult to evaluate and only a few Salem sets are known.

Descriptive Complexity

Theorem (joint with Alberto Marcone, Reimann and Manlio Valenti)

1. *The set of codes for closed Salem subsets of* $[0,1]$ *is* Π^0_3*-complete.*
2. *The set of real numbers* ξ *such that there is a Fourier measure making* ξ *random is* Σ^0_2*-complete*

- The proofs rely on compactness for the appropriate sets of measures.

Effective Fourier Dimension

Currently, there is no identified candidate for the Fourier dimension of a single real number ξ.

- The goal would be to identify

$$\sup\{s : \xi \text{ is 1-random for a Fourier measure for dimension } s.\}$$

  in terms of recursion theoretic properties of ξ.

Effective Fourier Dimension

Currently, there is no identified candidate for the Fourier dimension of a single real number ξ.

- The goal would be to identify

$$\sup\{s : \xi \text{ is 1-random for a Fourier measure for dimension } s.\}$$

  in terms of recursion theoretic properties of ξ.

Theorem (joint with Verónica Becher and Reimann, generalizes R. Baker (unpublished) and may have been known earlier)

Suppose that μ is a Fourier measure and that $(b_i : i \in \mathbb{N})$ is a sequence of distinct integers. Then, for μ-almost-every real ξ, $(b_i\xi : i \in \mathbb{N})$ is uniformly distributed mod 1.

Discrepancy

Definition

Let $\overrightarrow{x} = (\xi_n : n \in \omega)$ be a sequence of real numbers in $[0, 1]$. The discrepancy of $\overrightarrow{x}$ at N is

$$D(\overrightarrow{x}, N) = \sup_{0 \leq a < b \leq 1} \left| \frac{\#\{i : a \leq x_i \leq b\}}{N} - (b - a) \right| .$$

- The discrepancy of $\overrightarrow{x}$ measures how well and how quickly $\overrightarrow{x}$ distributes its elements as a function of N.

Discrepancy

Definition

Let $\overrightarrow{x} = (\xi_n : n \in \omega)$ be a sequence of real numbers in $[0, 1]$. The discrepancy of $\overrightarrow{x}$ at N is

$$D(\overrightarrow{x}, N) = \sup_{0 \leq a < b \leq 1} \left| \frac{\#\{i : a \leq x_i \leq b\}}{N} - (b - a) \right|.$$

- The discrepancy of $\overrightarrow{x}$ measures how well and how quickly $\overrightarrow{x}$ distributes its elements as a function of N.

Question

Suppose that μ is a Fourier measure for dimension s and that b is an integer greater than 1. What is the μ-almost-everywhere discrepancy function for $(b^n\xi : n \in \omega)$?

Diophantine Approximation

Émile Borel (1909): normal numbers

Definition

Let ξ be a real number.

- ξ is *simply normal to base b* if in its base-b expansion, $(\xi)_b$, each digit appears with limiting frequency equal to $1/b$.
- ξ is *normal to base b* if in $(\xi)_b$ every finite pattern of numbers occurs with limiting frequency equal to the expected value $1/b^\ell$, where ℓ is the pattern length.
  - Equivalently, $(b^n\xi : n \in \omega)$ is uniformly distributed mod 1.
- ξ is *absolutely normal* if it is normal to every base b.

Normality

analogous to randomness

Theorem (Borel (1909))

Almost all real numbers are simply normal in every base.

Normality

analogous to randomness

Theorem (Borel (1909))

Almost all real numbers are simply normal in every base.

Simply normal, normal and absolutely normal numbers play the role of random reals for meeting the full measure sets for simple normality for a base b, for powers of b, and for all b, respectively.

Normality

analogous to randomness

Theorem (Borel (1909))

Almost all real numbers are simply normal in every base.

Simply normal, normal and absolutely normal numbers play the role of random reals for meeting the full measure sets for simple normality for a base b, for powers of b, and for all b, respectively.

Analogous By a theorem of Schnorr and Stimm, the normality of a sequence is equivalent to its incompressibility by finite automata.

Normality

analogous to randomness

Theorem (Borel (1909))

Almost all real numbers are simply normal in every base.

Simply normal, normal and absolutely normal numbers play the role of random reals for meeting the full measure sets for simple normality for a base b, for powers of b, and for all b, respectively.

Analogous By a theorem of Schnorr and Stimm, the normality of a sequence is equivalent to its incompressibility by finite automata.

Disanalogous Unlike in recursion theory, the integer bases provide a one-parameter family of randomness criteria.

Normality: Depends on Base

Theorem (Cassels and Schmidt (1959))

There is a real number which is normal in one base and not in another.

Normality: Depends on Base

Theorem (Cassels and Schmidt (1959))

There is a real number which is normal in one base and not in another.

Theorem (joint with Becher and Yann Bugeaud (2013))

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integer m, if $b^m \in M$ then $b \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Irrationality Exponents

analogous to effective Hausdorff dimension

Definition

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Irrationality Exponents

analogous to effective Hausdorff dimension

Definition

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- When z is large, instances of $0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^z}$ are instances of algorithmic compression.
- (Jarník (1929) and Besicovitch (1934)) The set $\{\xi : \xi \text{ has irrationality exponent } \alpha\}$ has Hausdorff dimension $2/\alpha$.
- (Kaufman (1981)) The above set is a Salem set.

Normality and Irrationality Exponents

Theorem (Bugeaud (2002), based on the construction of Kaufman)

There is an absolutely normal Liouville number.

Normality and Irrationality Exponents

Disanalogous

Theorem (Bugeaud (2002), based on the construction of Kaufman)

There is an absolutely normal Liouville number.

A few years ago, we extended work of Amou and Bugeaud.

Theorem (joint with Becher)

Suppose $a \in [2, \infty]$ and M is a subset of the integers greater than or equal to 2 as above. Then there is a real number ξ such that ξ is simply normal to exactly the bases in M and ξ has exponent of irrationality a.

Irrationality Exponents Relative to Independent Bases

As with normality, the integer bases provide a one-parameter family of compressibility criteria.

Irrationality Exponents Relative to Independent Bases

As with normality, the integer bases provide a one-parameter family of compressibility criteria.

Definition (following Amou and Bugeaud (2010))

For a real number ξ, the *base-b irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{b^k}\right| < \frac{1}{\left(b^k\right)^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Irrationality Exponents Relative to Independent Bases

As with normality, the integer bases provide a one-parameter family of compressibility criteria.

Definition (following Amou and Bugeaud (2010))

For a real number ξ, the *base-b irrationality exponent of ξ* is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{b^k}\right| < \frac{1}{\left(b^k\right)^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- An application of Baker-Schmidt (1971):

$$\{\xi : \xi \text{ has base-}b \text{ irrationality exponent } \alpha\}$$

has Hausdorff dimension $1/\alpha$.

Irrationality Exponent: Depends on Base

Theorem (Amou and Bugeaud (2010))

Suppose that b_1 and b_2 are multiplicative independent bases, and suppose that a_2 and a_3 are greater than $1 + \frac{1+\sqrt{5}}{2}$. There is a real number whose base-b_1 and base-b_2 exponents of irrationality are a_2 and a_3, respectively.

- The proof relies on the theory of continued fractions. A measure theoretic approach would be welcome.

Random or Compressible: Depends on Base

Theorem

There is a real number ξ which is normal to base 2 and whose base 10 exponent of irrationality is equal to ∞.

- For every k there is an n such that the decimal expansion of ξ has a block of $k \cdot n$ zeros immediately following its kth digit.
- The proof uses a generalized version of Stoneham numbers, which rests upon 2's being prime and 10's being a product of 2 with another prime.

Random or Compressible: Depends on Base

Theorem

There is a real number ξ which is normal to base 2 and whose base 10 exponent of irrationality is equal to ∞.

- For every k there is an n such that the decimal expansion of ξ has a block of $k \cdot n$ zeros immediately following its kth digit.
- The proof uses a generalized version of Stoneham numbers, which rests upon 2's being prime and 10's being a product of 2 with another prime.

Question

Is there a real number which is normal to base-2 and has base-3 exponent of irrationality equal to ∞?

Finding Higher Recursion Theory