Polishable equivalence relations

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Outline of Topics

Polishable equivalence relations

Recall the notion of **uniformity** $\mathcal{V}$ on a set X:
$\mathcal{V}$ is a closed upwards family of symmetric sets, whose intersection is the diagonal, and such that for each $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ with

$$W \circ W \subseteq V.$$

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Weil: $t(\mathcal{V})$ is metrizable if and only if $\mathcal{V}$ has a countable basis.

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— Γ is $\mathcal{V}$-**locally dense**.

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Γ is **$\mathcal{V}$-locally dense** if, for each $V \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that, for each $x \in X$,

$$\{gx \colon g \in \Gamma,\, g \leq V\}$$

is $t(\mathcal{V})$-dense in W_x.

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— $t(\mathcal{V})$ is completely metrizable, and $[x]_E$ is G_δ with respect to $t(\mathcal{V})$;

— Γ is countable, and Γx is a $t(\mathcal{V})$-dense subset of $[x]_E$.

Relationships with other classes of equivalence relations

Larger classes of equivalence relations

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If E is a Polishable equivalence relation that is Borel, then E is idealistic.

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If E is a Polishable equivalence relation on a Polish space X as witnessed by $\mathcal{V}$ and Γ, then there exists a uniformity $\mathcal{V}'$ on X such that

— $\mathcal{V} \subseteq \mathcal{V}'$,

— *E is Polishable as witnessed by $\mathcal{V}'$ and Γ,*

— *$E \in \mathcal{V}'$, that is, $(E, \mathcal{V}')$ is a uniform equivalence relation.*

Smaller classes of equivalence relations

1. Orbit equivalence relations of Polish group actions

Let a be a continuous action of a Polish group G on a Polish space (X, σ).
Let E_a be the **orbit equivalence relation**, that is,

$$xE_ay \text{ iff } y = gx \text{ for some } g \in G.$$

The action a induces the uniformity

$$\mathcal{V}_a = \text{ the upward closure of } \{\hat{V} \colon 1 \in V = V^{-1} \text{ open in } G\},$$

where

$$\hat{V} = \{(x, y) \in X \times X \colon \exists g \in V \; gx = y\}.$$

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$\Gamma_a =$ the group of transformations of X induced by a countable dense subgroup of G.

The orbit equivalence relation E_a is Polishable as witnessed by $\mathcal{V}_a$ and Γ_a.

2. Isomorphism equivalence relation in continuous model theory, Ben Yaacov–Doucha–Nies–Tsankov, 2017

L a countable language containing only predicate symbols.

To each $P \in L$, L associates $n_P \in \mathbb{N}$, a closed interval I_P, and a function $\Delta_P \colon [0, \infty)^{n_P} \to [0, \infty)$.

L contains one distinguished binary symbol d.

A **structure** is a metric separable space A with d interpreted as the metric on A, and with $P \in L$ interpreted as a function $P^A \colon A^{n_P} \to I_P$ such that Δ_P is a modulus of uniform continuity for P^A.

Define

$$X(L) \subseteq \prod_{P \in L} \mathbb{R}^{\mathbb{N}^{n_P}}$$

to consist of x that code an L-structure, which we denote by $\widehat{x}$.

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The isomorphism equivalence relation E_L is Polishable.

Scott analysis for Polishable equivalence relations

(X, τ) a topological space

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An operation on subsets of $X \times X$
For $A \subseteq X \times X$ symmetric, let

$$A^\tau = \{(x, y) \in X \times X \colon y \in \overline{A_x} \text{ and } x \in \overline{A^y}\}.$$

A^τ is symmetric.

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An operation on families of symmetric subsets of $X \times X$
For an upward closed family $\mathcal{U}$ of symmetric subsets of $X \times X$, let

$$\mathcal{U}^\tau = \text{ the upward closure of } \{A^\tau \colon A \in \mathcal{U}\}.$$

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$$t_\xi = \sigma \vee \bigvee_{\gamma<\xi} t(U_\gamma) \text{ and } U_\xi = \mathcal{V}^{t_\xi}.$$

Denote t_ξ and U_ξ by

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$U_\xi(\sigma, \mathcal{V})$ is a uniformity and $t_\xi(\sigma, \mathcal{V})$ is a topology.

Additionally given: E an equivalence relation on X.

Define

$$xE_\xi(\sigma, \mathcal{V})y \text{ iff } \mathrm{cl}_{t_\xi(\sigma,\mathcal{V})}([x]_E) = \mathrm{cl}_{t_\xi(\sigma,\mathcal{V})}([y]_E).$$

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$E_\xi(\sigma,\mathcal{V})$ is an equivalence relation and

$$E_0(\sigma,\mathcal{V})\supseteq E_1(\sigma,\mathcal{V})\supseteq\cdots\supseteq E_\xi(\sigma,\mathcal{V})\supseteq\cdots\supseteq E.$$

Theorem

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(iii) $E_\xi(\sigma, \mathcal{V})$ *is* $\mathbf{\Pi}^0_{\iota(\xi)}$ *with respect to* $\sigma \times \sigma$;

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Moreover, for each $\xi < \omega_1$,

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$\iota(\xi) = 2 + \lambda(\xi) + 2n(\xi)$, where $\xi = \lambda(\xi) + n(\xi)$ with $\lambda(\xi)$ limit or 0 and $n(\xi) < \omega$.

Relationships with other Scott analyses

A relationship with Polishable subgroups, S. 1999

(H, σ) a Polish group, G a subgroup of H

$E_{H/G}$ = the orbit equivalence relation of the right action of G on H

$$G \times H \ni (g, h) \to hg^{-1} \in H.$$

So $h_1 E_{H/G} h_2 \iff h_1 G = h_2 G$.

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$G < H$ is called a **Polishable subgroup of** H if there exists a Polish group topology τ on G that contains σ restricted to G.

The Polish group (G,τ) acts continuously on the Polish space (H,σ).

$E_{H/G}$ is the orbit equivalence relation of this action.

So $E_{H/G}$ is Polishable as witnessed by $\mathcal{V}_G$ and Γ_G, and it admits a Scott analysis, that is, we can form the uniformities $U_\xi(\sigma,\mathcal{V}_G)$ (and topologies $t_\xi(\sigma,\mathcal{V}_G)$) and equivalence relations $E_\xi(\sigma,\mathcal{V}_G)$.

We write

$$t_\xi = t_\xi(\sigma,\mathcal{V}_G) \text{ and } (E_{H/G})_\xi = (E_{H/G})_\xi(\sigma,\mathcal{V}_G).$$

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(iv) *$G_{\xi+1}$ is a $\mathbf{\Pi}^0_3$ with respect to t_ξ subset of G_ξ;*

(v) *if $A \supseteq G$ is $\mathbf{\Pi}^0_3$ with respect to t_ξ, then $A \cap G_{\xi+1}$ is comeager in $G_{\xi+1}$ with respect to $t_{\xi+1}$.*

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Moreover,

(vi) *if G is in $\bigcup_{\gamma<\xi} \mathbf{\Pi}^0_{1+\gamma}$, then $G = G_\xi$.*

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(i) $E_a = \bigcap_{\xi<\omega_1} E_\xi$;

(ii) each equivalence class C of E_ξ carries a topology with respect to which it contains a dense Polish subspace;

(iii) E_ξ is $\mathbf{\Pi}^0_{\xi+n}$ for some $n < \omega$;

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Question 1. Is Hjorth's analysis for orbit equivalence relations of continuous Polish group actions equal to the analysis produced here?

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Filtrations of topologies

Let $\sigma \subseteq \tau$ be topologies on the same set.

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A sequence $(\tau_\xi)_\xi$ of topologies is a **filtration from σ to τ** if

$$\sigma = \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_\xi \subseteq \cdots \subseteq \tau$$

and, for each α, if F is τ_ξ-closed for some $\xi < \alpha$, then

$$\mathrm{int}_{\tau_\alpha}(F) = \mathrm{int}_\tau(F).$$

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Assume that τ has a neighborhood basis consisting of sets that are C-sets with respect to σ.
If $\tau_{\xi_0} = \tau_{\xi_0+1}$, then $\tau_{\xi_0} = \tau$.

Theorem

Let $\sigma \subseteq \tau$ be topologies, with τ being regular and Baire, and let $\alpha \leq \omega_1$ be a limit ordinal.
Assume that $(\tau_\xi)_{\xi<\alpha}$ is a filtration from σ to τ, with τ_ξ completely metrizable for each $\xi < \alpha$.

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If τ has a neighborhood basis consisting of sets that are in $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_{1+\xi}$ with respect to σ, then

$$\tau = \bigvee_{\xi<\alpha} \tau_\xi.$$

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Corollary

Fix $\alpha \leq \omega_1$. Let τ be Baire, and let τ_ξ, for $\xi < \alpha$, be completely metrizable. Assume E equivalence classes are τ-open.

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Corollary

Fix $\alpha \leq \omega_1$. Let τ be Baire, and let τ_ξ, for $\xi < \alpha$, be completely metrizable. Assume E equivalence classes are τ-open.
If all E equivalence classes are in $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_{1+\xi}$ with respect to σ, then $E = \bigcap_{\xi<\alpha} E_\xi$.