# Decision Problems in Borel Combinatorics

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# Bernoulli Shift and Cayley Graph

For a countable group $G$, the Bernoulli shift on $G$ is the dynamical system $\cdot : G \times 2^G \to 2^G$ defined by

$$(g \cdot x)(h) = x(g^{-1}h).$$

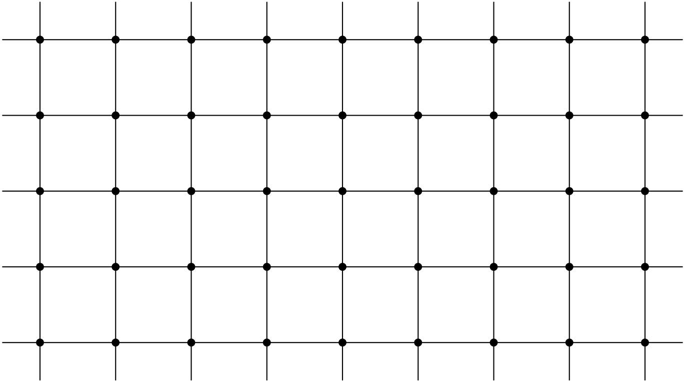
When $G = \langle S \rangle$ for a finite $S = S^{-1}$, one can define a Cayley graph on $2^G$ by

$$(x, y) \in E(2^G) \iff \exists s \in S \; s \cdot x = y.$$

Examples $2^{\mathbb{Z}^n}$, $F(2^{\mathbb{Z}^2})$, $2^{\mathbb{F}_n}$, $F(2^{\mathbb{F}_n})$

# Bernoulli Shift and Cayley Graph

The Cayley graph on $F(2^{\mathbb{Z}^2})$ consists of continuum many components, with each component a grid resembling $\mathbb{Z}^2$.



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A $\mathbb{Z}^n$-subshift of finite type is a $Y \subseteq b^{\mathbb{Z}^n}$ for which there is a finite set $\{p_1, \ldots, p_n\}$ of patterns such that

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We denote such a subshift as $Y_{b,\vec{p}}$.

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This problem is equivalent to the instances of the Graph Homomorphism Problem:

Question Is there a continuous (or Borel) graph homomorphism from $F(2^{\mathbb{Z}^n})$ into $K_3$ or $K_4$?

Answer The continuous chromatic number of $F(2^{\mathbb{Z}^n})$ is 4, and the Borel chromatic number of $F(2^{\mathbb{Z}^2})$ is 3.

# Examples

There is no continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ into the Petersen graph.

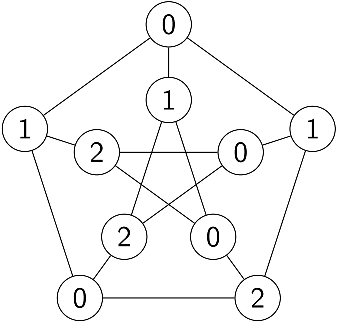


Figure: The Petersen graph with a three-coloring.

# Examples

There is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ into the Grötzsch graph.

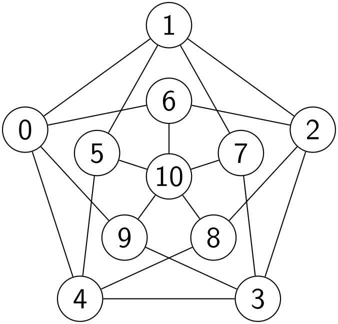


Figure: The Grötzsch Graph. The odd cycle $\gamma = (0, 1, 2, 3, 9, 0)$ has order 2 in the homotopy group modded out by four-cycles.

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Main Theorem This is a $\Sigma^0_1$-complete problem.

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Theorem There are *finite* graphs $\Gamma_{n,p,q}$, for each triple $(n,p,q)$ of positive integers with $n < p, q$, such that for all finite graphs $\Gamma$ the following are equivalent:

1. there is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to $\Gamma$;
2. there is a graph homomorphism from $\Gamma_{n,p,q}$ to $\Gamma$ for some $n < p, q$ with $(p,q) = 1$;
3. for all $n$ and sufficiently large $p, q$ with $(p,q) = 1$, there is a graph homomorphism from $\Gamma_{n,p,q}$ to $\Gamma$.

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- 4 commutativity tiles
- 2 long horizontal tiles
- 2 long vertical tiles

# Torus Tiles

| | $n$ | | $p$ |
|---|---|---|---|
| $n$ | $R_\times$ | $R_c$ | $R_\times$ |
| | $R_a$ | | $R_a$ |
| $p$ | $R_\times$ | $R_c$ | $R_\times$ |

$G_{ca=ac}$

| | $n$ | | $p$ |
|---|---|---|---|
| $n$ | $R_\times$ | $R_c$ | $R_\times$ |
| | $R_b$ | | $R_b$ |
| $q$ | $R_\times$ | $R_c$ | $R_\times$ |

$G_{cb=bc}$

$$R_\times : n \times n, \;\; R_a : n \times (p-n), \;\; R_b : n \times (q-n)$$
$$R_c : (p-n) \times n, \;\; R_d : (q-n) \times n$$

# Torus Tiles (continued)

Columns: $n$, $q$; rows: $n$, $p$

| | | |
|---|---|---|
| $R_\times$ | $R_d$ | $R_\times$ |
| $R_a$ | | $R_a$ |
| $R_\times$ | $R_d$ | $R_\times$ |

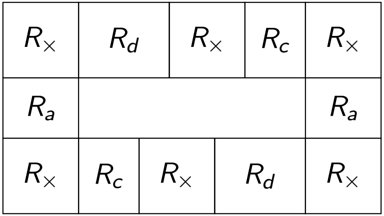
$G_{da=ad}$

Columns: $n$, $q$; rows: $n$, $q$

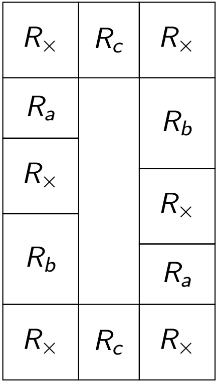
| | | |
|---|---|---|
| $R_\times$ | $R_d$ | $R_\times$ |
| $R_b$ | | $R_b$ |
| $R_\times$ | $R_d$ | $R_\times$ |

$G_{db=bd}$

# Commutativity Tiles

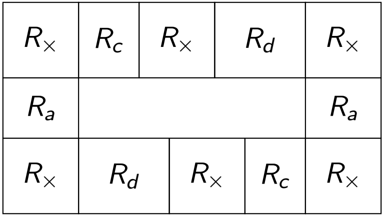



$G_{dca=acd}$

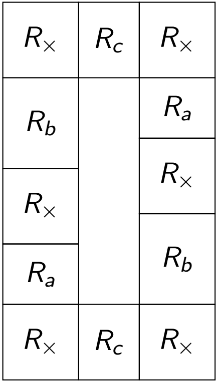



$G_{cba=abc}$

# Commutativity Tiles (continued)



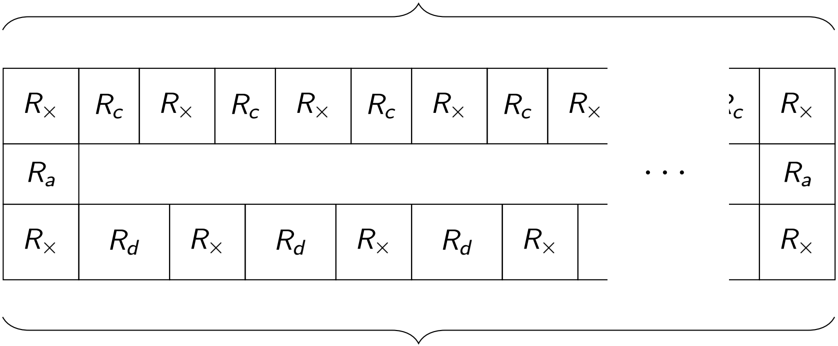

$G_{cda=adc}$




$G_{cab=bac}$

# Long Horizontal Tiles

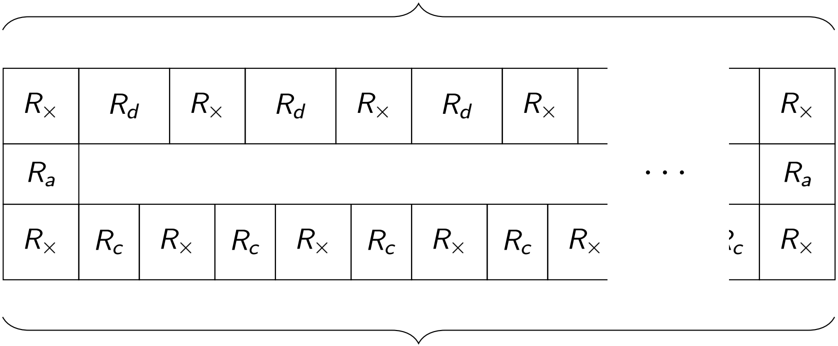
$q$ copies of $R_c$, $q+1$ copies of $R_\times$



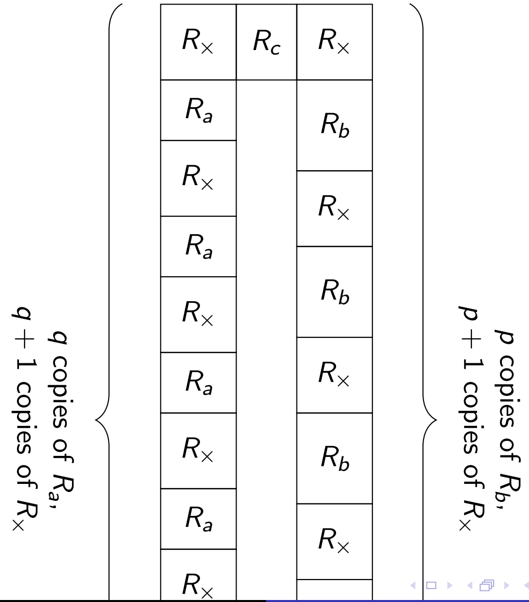

$p$ copies of $R_d$, $p+1$ copies of $R_\times$

# Long Horizontal Tiles (continued)

$p$ copies of $R_d$, $p+1$ copies of $R_\times$




$q$ copies of $R_c$, $q+1$ copies of $R_\times$


$R_\times$
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$q$ copies of $R_a$,
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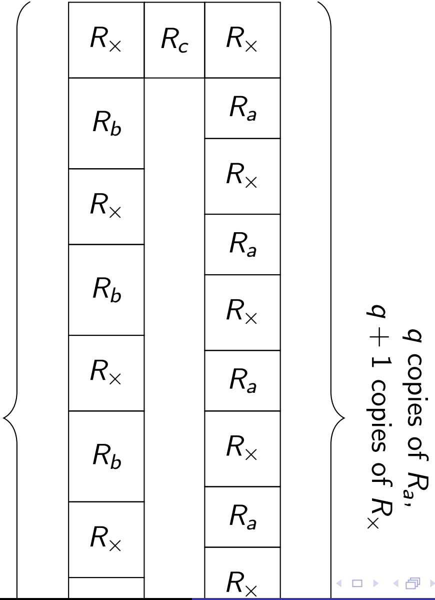
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$R_\times$
$q$ copies of $R_a$,
$q+1$ copies of $R_\times$

The smallest 12-tiles graph is $\Gamma_{1,2,3}$: 60 nodes and 180 edges.

# Undecidability of the Continuous Graph Homomorphism Problem

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Theorem The set of all finite $\Gamma$ for which is there a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to $\Gamma$ is not computable.

There is not a computable bound of how large $p$ and $q$ will be for the first $\Gamma_{n,p,q}$ to admit a graph homomorphism to a given finite $\Gamma$.

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We define a computable reduction of this word problem to the Continuous Graph Homomorphism Problem for $F(2^{\mathbb{Z}^2})$.

# Undecidability of the Continuous Graph Homomorphism Problem

Start with a finite presentation

$$G = \langle a_1, \ldots, a_k \mid r_1, \ldots, r_l \rangle$$

of a torsion-free group $G_n$, and

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of a torsion-free group $G_n$, and

$$\text{a distinguished word } w = w(a_1, \ldots, a_k).$$

(*) There is (a lower bound) $\alpha > 0$ such that, if the distinguished word $w \neq e$ in $G$, then for all integer $m \geq 1$, $w^m$ is not equal in $G_n$ to any word of length $\leq \alpha m$.

# Undecidability of the Continuous Graph Homomorphism Problem

Consider

$$G' = \langle a_1, \dots, a_k, z | r_1, \dots, r_l, z^2 w^{-1} \rangle = \langle a_1, \dots, a_{k+1} \,|\, r_1, \dots, r_l, r_{l+1} \rangle.$$

Construct a graph $\Gamma'$. $\Gamma'$ will have a distinguished vertex $v_0$. For each of the generators of $G'$, we add a sufficiently long cycle $\beta_i$ of length $\ell_i > 4$ that starts and ends at the vertex $v_0$. We make the edge sets of these cycles pairwise disjoint. This gives a natural notion of length $\ell(a_i) = \ell_i$ which extends in the obvious manner to reduced words in the free group generated by the $a_i$. For each word $r_j$, we wish to add to $\Gamma'$ a rectangular grid-graph $R_j$ whose length and width are both $> 4$ and whose perimeter is equal to $\ell(r_j)$. In order for this to be possible, we will need to make certain that each $\ell(r_j)$ is a large enough even number.

The edges used in the various $R_j$ are pairwise disjoint, and are disjoint from the edges used in the cycles corrresponding to the generators $a_i$. We then label the edges (say going clockwise, starting with the upper-left vertex) of the boundary of $R_j$ with the edges occurring in the concatenation of the paths corrresponding to the generators in the word $r_j$.
Finally, $\Gamma$ is obtained from $\Gamma'$ by forming the quotient graph where vertices on the perimeters of the $R_j$ are identified with the corresponding vertex in one of the $a_i$.

Instead of using the Twelve Tiles Theorem directly, the proof uses some corollaries of the Twelve Tiles Theorem that give positive and negative conditions in terms of the homotopy group of the graph $\Gamma$.

Theorem If there is an odd-length cycle $\gamma$ which has finite order in $\pi_1^*(\Gamma)$, then there is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to $\Gamma$.

Theorem Suppose for every $n$ there are $p, q > n$ with $(p, q) = 1$ such that, for any $p$-cycle $\gamma$ in $\Gamma$, $\gamma^q$ is not a $p$-th power in $\pi_1^*(\Gamma)$. Then there is no continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to $\Gamma$.

# What about $F(2^{\mathbb{Z}})$?

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Theorem The Continuous Graph Homomorphism Problem for $F(2^{\mathbb{Z}})$ is decidable.

# Summary

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For $n = 2$, the continuous Subshift Problem and the continuous Graph Homomorphism Problem are $\Sigma^0_1$-complete.

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For $n = 2$, the continuous Subshift Problem and the continuous Graph Homomorphism Problem are $\Sigma_1^0$-complete.

All other cases are open.

# Thank You!