Choiceless Set-theoretic Geology

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Abstract

Set-theoretic geology is a study of the structure of all ground models of the universe V. We will try to extend the standard set-theoretic geology to choiceless set-theoretic geology, which can treat with choiceless models.

Definability of ground models

Throughout this talk, forcing means a set forcing.

Fact (Laver, Woodin)

In any forcing extension $V[G]$ of V, the universe V is a (first order) definable class in $V[G]$ with some parameters from V.

Changing the viewpoint

Every ground model of V is definable by some first order formula of set-theory.

A ground is a ground model of V.

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Every ground model of V is definable by some first order formula of set-theory.

A ground is a ground model of V.

Uniform definability of grounds

Actually all grounds can be defined by some uniform way.

Fact (Fuchs-Hamkins-Reitz)

There is a first order formula $\varphi(x, y)$ of set theory such that:

1. *For each set r, the class $W_r = \{x \mid \varphi(x, r)\}$ is a ground of V with $r \in W_r$ ($W_r = V$ is possible).*
2. *For every transitive model $M \subseteq V$ of ZFC, if M is a ground of V, then there is r with $M = W_r$.*

Remark

The formula φ does not depend on V: In any grounds and generic extensions of V, φ defines its grounds by the same way.

Set-theoretic geology

The uniform definability allows us to study the structure of the collection of grounds $\{W_r \mid r \in V\}$ in ZFC: e.g.,

- One can define (in ZFC) the intersection of two grounds.
- One can ask (in ZFC) whether $\forall r \exists s \, (W_s \subsetneq W_r)$?

This study is now called set-theoretic geology.

Set-theoretic geology without AC

- In the standard geology, the universe V and all grounds are supposed to satisfy the Axiom of Choice (AC).
- However it is possible that V has a ground which does not satisfy AC.
- Moreover choiceless context is very common (e.g., ZF+AD), and forcing over choiceless model is very useful (e.g., $\mathbb{P}_{max}$ over $L(\mathbb{R})$).
- So it is natural to extend the standard set-theoretic geology to Choiceless Set-theoretic geology.

Question

Can we develop choiceless set-theoretic geology?

- A first problem is the definability of grounds of V.

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Covering and approximation properties

In ZFC, Hamkins' covering and approximation properties are important tools for proving the (uniform) definability of grounds.

Definition (Hamkins)

Let κ be a cardinal, and $M \subseteq V$ a transitive (set or class) model of ZFC.

1. M satisfies the κ-covering property if for every $a \in [\mathrm{ON}]^{<\kappa}$ there exists $b \in M \cap [\mathrm{ON}]^{<\kappa}$ with $a \subseteq b$.
2. M satisfies the κ-approximation property if for every set A of ordinals, $\forall x \in [ON]^{<\kappa} \cap M (A \cap x \in M)$ then $A \in M$.

Fact (Hamkins)

Let M, N be transitive models of ZFC. If M and N satisfy the κ-covering and the κ-approximation properties, and $\mathcal{P}(\kappa^+) \cap M = \mathcal{P}(\kappa^+) \cap N$, then $M \cap \mathcal{P}(ON) = N \cap \mathcal{P}(ON)$, hence $M = N$.

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Fact (Hamkins)

Every ground satisfies the κ-covering and approximation properties for some κ.

- In the proof of the uniform definability of grounds, the property $M \cap \mathcal{P}(ON) = N \cap \mathcal{P}(ON) \Rightarrow M = N$ is essential. However this is not valid if AC fails in M and N.
- Moreover, it is not clear that, in ZF, every ground satisfies the covering and approximation properties.

Question (in ZF, open)

1. For every poset $\mathbb{P}$ and generic G, is V definable in $V[G]$?
2. Are all (choiceless) grounds of V uniformly definable?

Fact (Gitman-Johnstone, in ZF)

Suppose DC_κ holds. Then for every poset $\mathbb{P}$ with size $\leq \kappa$ ($\mathbb{P}$ is assumed to be well-orderable), V is definable in $V^{\mathbb{P}}$ with some parameters.

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Convention

- From now on, our base theory is ZF unless otherwise specified.
- A ground is a transitive model of ZF such that V is a set-generic extension of it.

Löwenheim-Skolem property in ZF

For the uniform definability of grounds, we give a partial answer: It is possible if some Löwenheim-Skolem property holds.

Definition

We say that κ is a Löwenheim-Skolem cardinal if for every $\alpha > \kappa$, $p \in V_\alpha$, and $\gamma < \kappa$, there is $\beta > \alpha$ and $X \prec V_\beta$ such that:

1. $p \in X$ and $V_\gamma \subseteq X$.
2. ${}^{V_\gamma}(X \cap V_\alpha) \subseteq X$.
3. The transitive collapse of X belongs to V_κ.

Remark

- In ZFC, κ is Löwenheim-Skolem if and only if $\beth_\kappa = \kappa$, so there are always proper class many Löwenheim-Skolem cardinals.
- If κ is a limit of Löwenheim-Skolem cardinals, then κ is Löwenheim-Skolem as well.

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Some consequences of LS property

The existence of a Löwenheim-Skolem cardinal is not provable from ZF:

Lemma

Suppose κ is a Löwenheim-Skolem cardinal and $\lambda \geq \kappa$ a cardinal. Then the following hold:

1. *For every $\alpha < \kappa$, there is no cofinal map from V_α onto λ^+. Hence $\mathrm{cf}(\lambda^+) \geq \kappa$.*
2. *For every $\alpha < \kappa$ and family $\{S_x \mid x \in V_\alpha\}$ of non-stationary sets in λ^+, the union $\bigcup_{x \in V_\alpha} S_x$ is non-stationary in λ^+. Hence the non-stationary ideal over λ^+ is κ-complete.*

In Gitik's model with no regular uncountable cardinals, Löwenheim-Skolem cardinal does not exist.

Corollary

If κ is a singular Löwenheim-Skolem cardinal, (e.g., singular limit of Löwenheim-Skolem cardinals), then κ^+ is regular, and the non-stationary ideal over κ^+ is κ^+-complete. In addition for every regressive function $f : \kappa^+ \to \kappa^+$, there is $\gamma < \kappa^+$ such that the set $\{\alpha < \kappa \mid f(\alpha) = \gamma\}$ is stationary in κ^+.

Note that:

Fact (Woodin)

If κ is a singular limit of supercompact cardinals, then κ^+ is regular, and the non-stationary ideal over κ^+ is κ^+-complete.

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Large cardinal yields LS property

Definition (Woodin)

An uncountable cardinal κ is supercompact if for every $\alpha > \kappa$, there is $\beta > \alpha$, a transitive set X, and an elementary embedding $j : V_\beta \to X$ such that the critical point of j is κ, $\alpha < j(\kappa)$, and ${}^{V_\alpha}X \subseteq X$.

Lemma

1. *Every supercompact cardinal is Löwenheim-Skolem, and a limit of Löwenheim-Skolem cardinals.*
2. *Hence if there are proper class many supercompact cardinals, then there are proper class many Löwenheim-Skolem cardinals as well.*

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Uniform definability of grounds

Under the assumption that there are proper class many Löwenheim-Skolem cardinals, we can define all grounds uniformly.

Theorem

Suppose there are proper class many Löwenheim-Skolem cardinals. Then all grounds are uniformly definable: There is a first order formula $\varphi(x, y)$ of set-theory such that:

1. For each set r, the class $W_r = \{x \mid \varphi(x, r)\}$ is a ground of V with $r \in W_r$.
2. For every ground $M \subseteq V$, there is r with $M = W_r$.

In particular, if there are proper class many supercompact cardinals, then all grounds are uniformly definable.

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Coarse measure

Let us outline the proof of the uniform definability of grounds. For the sake of this, we introduce alternative measure on sets, which is coarser than the standard cardinality but works without AC.

Definition

For a set x, let $||x||$, the norm of x, is the least ordinal α such that there is a surjection from V_α onto x.

Remark

1. $||x|| \leq \mathrm{rank}(x)$.
2. If $x \subseteq y$ then $||x|| \leq ||y||$.
3. If $M \subseteq V$ is a transitive model of (a sufficiently large fragment of) ZF and $x \in M$, then $||x|| \leq ||x||^M$.
4. κ is Löwenheim-Skolem $\iff$ every frist order structure has an elementary substructure with some closure property and norm $< \kappa$.

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Covering and approximation properties

Definition

Let M be a transitive (set or class) model of (a sufficiently large fragment of) ZF, and α an ordinal with $\alpha \in M$.

1. M satisfies the α-norm covering property if for every $\beta \in M$ and $x \subseteq M_\beta$, if $||x|| < \alpha$ then there is $y \in M$ such that $x \subseteq y$ and $||y||^M < \alpha$.
2. M satisfies the α-norm approximation property if for every $\beta \in M$ and $A \subseteq M_\beta$, if $A \cap x \in M$ for every $x \in M$ with $||x||^M < \alpha$, then $A \in M$.

Uniqueness of models with covering and approximation properties

Theorem

Let κ be a Löwenheim-Skolem cardinal, and M and N be transitive models of (a sufficiently large fragment of) ZF. If

1. $\kappa \in M \cap N$ and $M \cap \mathrm{ON} = N \cap \mathrm{ON}$.
2. M and N satisfy the γ-norm covering and approximation properties for some $\gamma < \kappa$.
3. $M_\kappa = N_\kappa$.

Then $M = N$.

By induction on the rank of sets.

Lemma

Suppose κ is a Löwenheim-Skolem cardinal. For every ground M of V, if there is a poset $\mathbb{P} \in M_\kappa$ such that V is a generic extension of M via $\mathbb{P}$, then M satisfies the κ-norm covering and approximation properties.

Theorem (Uniform definability of all grounds)

Suppose there are proper class many Löwenheim-Skolem cardinals. Then there is a first order formula $\varphi(x, y)$ of set theory such that:

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Generic absoluteness of LS property

Lemma

Suppose $M \subseteq V$ is a ground, κ a cardinal, and $V = M[G]$ for some $G \subseteq \mathbb{P} \in M_\kappa$.

1. *If κ is a limit of Löwenheim-Skolem cardinals, then κ is a Löwenheim-Skolem cardinal in M.*
2. *In M, if κ is a limit of Löwenheim-Skolem cardinals, then κ is Löwenheim-Skolem cardinal in V.*

Corollary

The statement "there are proper class many Löwenheim-Skolem cardinals" is absolute between V, all grounds, and all generic extensions.

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Mantle

Now suppose there are proper class many Löwenheim-Skolem cardinals, and φ defines all grounds. Then, in any grounds and generic extensions, φ defines its grounds by the same way.

Definition

The mantle $\mathbb{M}$ is the intersection of all grounds. The generic mantle $g\mathbb{M}$ is the intersection of all grounds of all generic extensions.

$\mathbb{M}$ and $g\mathbb{M}$ are parameter free definable transitive classes contain all ordinals. Clearly $g\mathbb{M} \subseteq \mathbb{M}$.

Fact (Fuchs-Hamkins-Reitz, U., in ZFC)

The mantle (the intersection of all grounds satisfying AC) is a model of ZFC, and it coincides with the generic mantle.

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In ZF, one can check that the generic mantle is a model of ZF.

Question

Is the mantle a model of ZF (or ZFC)? Does the mantle coincide with the generic mantle?

In ZFC, key property was the downward directedness of grounds.

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DDG may fail

However the downward dirctedness of grounds can fail in ZF.

Lemma

It is consistent that V satisfies AC, but V has two grounds M and N which have no common ground.

Suppose $V = L$, and let $\mathbb{P} = Fn(\omega_1, 2)$, adding ω_1-many Cohen reals. For a generic $G \times H \subseteq \mathbb{P} \times \mathbb{P}$, it is known that AC fails in $M = L(\mathbb{R})^{V[G]}$ and $N = L(\mathbb{R})^{V[H]}$, in addition $V[G]$ is a common generic extension of M and N.

By Solovay's theorem, we have $V[G] \cap V[H] = V = L$, hence $M \cap N = L$. But L is not a common ground of M and N, since L satisfies AC but M and N not. Hence $V[G]$ have two grounds which have no common ground.

Remark

The mantle and the generic mantle of M (and N) is just L, a model of *ZFC*.

If AC is forceable

In ZFC, there are proper class many Löwenheim-Skolem cardinals.

Corollary

If there is a poset which forces AC (AC is forceable), then there are proper class many Löwenheim-Skolem cardinals, and all grounds are uniformly definable.

Hence a natural question arises: Under what circumstances is AC forceable?

Fact (Blass)

The following are equivalent:

1. AC is forceable.
2. There is a set X such that for every set Y, there is an ordinal α and a surjection from $X \times \alpha$ onto Y.

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2. There is a set X such that for every set Y, there is an ordinal α and a surjection from $X \times \alpha$ onto Y.

If AC is forceable

In ZFC, there are proper class many Löwenheim-Skolem cardinals.

Corollary

If there is a poset which forces AC (AC is forceable), then there are proper class many Löwenheim-Skolem cardinals, and all grounds are uniformly definable.

Hence a natural question arises: Under what circumstances is AC forceable?

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Another characterization

For a transitive model $M \subseteq V$ of ZF and set X, let $M(X)$ be the minimal transitive model of ZF containing $M \cup \{X\}$.

Theorem

The following are equivalent:

1. There is a poset which forces AC.
2. There is a definable transitive model W of ZFC and a set X such that $W(X) = V$.
3. There is a definable transitive model W of ZFC and a set X such that $W(X) = V$, and W is a ground of some generic extension of V.

AC is forceable $\iff$ V is a small extension of a model of ZFC.

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Proof

Suppose $\mathbb{P}$ forces AC. Take a $(V, \mathbb{P} \times \mathbb{P})$-generic $G \times H$. The extensions $V[G \times H]$, $V[G]$, and $V[H]$ are models of ZFC, and $V[G]$, $V[H]$ are grounds of $V[G \times H]$. By the downward directedness of grounds, $V[G]$ and $V[H]$ have a common ground W satisfying AC. By Solovay's theorem, we have $V[G] \cap V[H] = V$. Then $W \subseteq V[G] \cap V[H] = V$, and one can check that W is definable in V.

Fact (Grigorieff)

Let M and N be transitive models of ZF, $M[G]$ a generic extension of M, and suppose $M \subseteq N \subseteq M[G]$. Then the following are equivalent:

1. N is a ground of $M[G]$.
2. There is a set $X \in N$ such that $N = M(X)$.

We know $V[G]$ is a generic extension of W and $W \subseteq V \subseteq V[G]$. By Grigorieff's theorem, there is a set $X \in V$ with $V = W(X)$.

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We know $V[G]$ is a generic extension of W and $W \subseteq V \subseteq V[G]$. By Grigorieff's theorem, there is a set $X \in V$ with $V = W(X)$.

Corollary

Suppose AC is forceable.

1. Every ground of V is of the form $M(X)$ for some definable transitive model M of ZFC and set X, and M is a ground of some generic extension of V.
2. For every two grounds M and N, there is $W \subseteq M \cap N$ which is a ground of some generic extension of V (but W may not be a ground of V, M, and N).
3. The generic mantle is a model of ZFC.

Corollary

Suppose V satisfies AC, and $W \subseteq V$ is a transitive model of ZF. Then the following are equivalent:

1. W is a ground of V.
2. There is a ground M of V such that M satisfies AC and $W = M(X)$ for some $X \in M$.

In particular,

- The family of grounds satisfying AC is dense in all grounds, with respects to $\subseteq$.
- The mantle (the intersection of all grounds which may or may not satisfy AC) is a model of ZFC, and coincides with the generic mantle.
- If V is a generic extension of the mantle $\mathbb{M}$, then every ground is of the form $\mathbb{M}(X)$ for some X.

Questions: Definability of grounds

Question

In ZF, are all grounds always uniformly definable?

If not, AC cannot be forceable over the universe. Some known candidates:

- A model with no uncountable regular cardinals (Gitik).
- A model which has proper class many infinite but Dedekind-finite sets (Monro).
- A model in which Fodor's lemma fails everywhere, and every non-stationary ideal is not σ-complete (Karagila).
- ...
- Bristol model M, which is a transitive model of ZF with $L \subseteq M \subseteq L^{\mathbb{C}}$, definable in $L^{\mathbb{C}}$, but $M \neq L(X)$ for every set X. AC is not forceable over M.

Questions: Geology of specific models

- A model with no uncountable regular cardinals (Gitik).
- A model which has a proper class many infinite but Dedekind-finite sets (Monro).
- A model in which Fodor's lemma fails everywhere, and every non-stationary ideal is not σ-complete (Karagila).
- Bristol model M, which is a model of ZF with $L \subseteq M \subseteq L^{\mathbb{C}}$, but $M \neq L(X)$ for every set X. AC is not forceable over M.

Test questions

1. Are grounds of these models uniformly definable? Or Do these models have proper class many Löwenheim-Skolem cardinals?
2. Does these models have a proper ground?
3. What is the mantle of these models?

Questions: AC-conjecture

If AC is forceable, then the geology of V is trivial: it is almost the same to the geology with AC. So we want to know non-trivial situation.

Question

Is it consistent that there are proper class many Löwenheim-Skolem cardinals, but AC is never forceable?

Theorem (Woodin, in essence)

1. If there is a Löwenheim-Skolem cardinal, then DC is forceable.
2. If there are proper class many regular Löwenheim-Skolem cardinals, then there is a class forcing which forces AC.

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This question may be connected with Woodin's AC-Conjecture.

Conjecture (Woodin)

If there are large cardinals, then AC is forceable.

Fact (Woodin, Under HOD-Conjecture)

If there are large cardinals, then there is a definable transitive model M of ZFC and a poset $\mathbb{P}$ such that every set of ordinals is generic over M via the poset $\mathbb{P}$.

Questions: Geological

There are many geological questions.

Question

1. Is the mantle a model of ZF?
2. Is it consistent that the mantle is a model ZF but not ZFC? How is the generic mantle?
3. Is it consistent that the mantle does not coincide with the generic mantle?

Fact (Fuchs-Hamkins-Reitz, in ZFC)

The universe can be the mantle of some class forcing extension: There is a class forcing extension $V[G]$ of V such that $V = \mathbb{M}^{V[G]}$.

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Does ZF-version of Fuchs-Hamkins-Reitz's theorem hold?

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Questions: Geological

Fact (Fuchs-Hamkins-Reitz, in ZFC)

1. *V has a class forcing extension $V[G]$ such that $V[G]$ has no proper ground, so $V[G] = \mathbb{M}^{V[G]}$.*
2. *It is consistent that V is not a forcing extension of $\mathbb{M}$.*

Fact (U., in ZFC)

If there exists an extendible cardinal, then the mantle is the minimum ground of V, hence V is a forcing extension of the mantle.

Question

In ZF, do similar results hold?

Questions: DDG

Lemma

It is consistent that there are set-many grounds $\{W_r \mid r \in X\}$ of V such that, no ground W of V with $W \subseteq \bigcap_{r \in X} W_r$.

However, in this model, we can find a model M such that $M \subseteq \bigcap_{r \in X} W_r$ and M is a generic ground, a ground of some generic extension of V.

Question

Does the following weak form of DDG hold?

For every set-many grounds $\{W_r \mid r \in X\}$ of V, there is a generic ground M with $M \subseteq \bigcap_{r \in X} W_r$.

If AC is forceable then this weak DDG holds.

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Thank you for your attention!