Uniform Martin's conjecture, locally

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Higher Recursion Theory and Set Theory

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In this talk:

$$\text{local} = \text{on a degree}$$

$$\text{global} = \text{on a cone}$$

Outline

Uniform Martin's conjecture, part I (proved by Slaman and Steel, 1988)

Assume AD. If $f : 2^\omega \to 2^\omega$ is UTI on a cone, then

- either $f(x) \geq_T x$ on a cone
- or f is constant on a cone (up to $\equiv_T$).

- We are proving a local version of this theorem
- we are deducing the original (global) version from the local one, assuming just TD instead of AD

As a consequence:

Corollary

The following are equivalent over ZF+DC:

- TD
- for every $f : 2^\omega \to 2^\omega$ which is UTI on a cone, either $f(x) \geq_T x$ on a cone, or f is constant on a cone.

So what about uniform Martin's conjecture, part II?

Uniform Martin's conjecture, part II (proved by Steel, 1983)

Assume AD. The set

$$J = \{ f \colon 2^\omega \to 2^\omega \mid f \text{ is UTI and } f(x) \geq_T x \text{ on a cone} \}$$

is pre-well-ordered by $\leq_M$. Moreover, there are no $f, g \in J$ such that

$$f <_M g <_M f'.$$

Question

- Does this theorem arise locally, too?
- Is it provable from TD?

There would be remarkable metamathematical consequences.

Introduction

$(2^\omega; \leq_T)$ induces on any Turing degree $\boldsymbol{x}$ a trivial structure.

$$\odot_T = \{\, (e, x, y) \in \omega \times 2^\omega \times 2^\omega \mid \varphi_e^x = y \,\}\,.$$

$$(e, x, y) \in (\odot_T) \iff y \leq_T x \textbf{ via } e.$$

We call $\odot_T$ "**Turing reducibility via**".

$(2^\omega, \omega; \odot_T)$ induces on $\boldsymbol{x}$ the structure $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$, where

$$\odot_{T,\boldsymbol{x}} = \{\, (e, x, y) \in \omega \times \boldsymbol{x} \times \boldsymbol{x} \mid \varphi_e^x = y \,\}\,.$$

This structure is non-trivial.

Macrocosm vs microcosm

Each Turing degree can be viewed

- as an "atom in the universe"
- as an "universe itself"

- How complicated is $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$?
- How does the complexity of $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$ vary when we vary $\boldsymbol{x}$?

Question

Is it the case that $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$ is embeddable into $(\boldsymbol{y}, \omega; \odot_{T,\boldsymbol{y}})$ iff $\boldsymbol{x} \leq_T \boldsymbol{y}$?

Answer

Yes!

Theorem 1

For all Turing degrees $\boldsymbol{x}$ and $\boldsymbol{y}$, the following are equivalent:

1. $\boldsymbol{x} \leq_T \boldsymbol{y}$
2. there is an embedding from $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$ into $(\boldsymbol{y}, \omega; \odot_{T,\boldsymbol{y}})$
3. there is a non-constant UTI function $f : \boldsymbol{x} \to \boldsymbol{y}$.

However...

the analogous fact for arithmetic degrees is false: there are arithmetic degrees $\boldsymbol{x}$ and $\boldsymbol{y}$ such that there is an embedding from $\boldsymbol{x}$ into $\boldsymbol{y}$ but $\boldsymbol{x} \not\leq_A \boldsymbol{y}$.

Embeddings

Let $X, Y \subseteq 2^\omega$ and let

$$(\odot_{T,X}) = (\odot_T) \cap (\omega \times X \times X)$$

Definition

An embedding from $(X, \omega; \odot_{T,X})$ to $(Y, \omega; \odot_{T;Y})$ is a pair (f, u) where

1. $f : X \to Y$
2. $u : \omega \to \omega$
3. (f, u) preserves the truth of atomic formulas in *both* directions, that is:
   - a $x_1 = x_2 \iff f(x_1) = f(x_2)$ (i.e. f is injective)
   - b $e_1 = e_2 \iff u(e_1) = u(e_2)$ (i.e. u is injective)
   - c $(e, x_1, x_2) \in (\odot_{T,X}) \iff (u(e), f(x_1), f(x_2)) \in (\odot_{T,Y})$

Morphisms

Definition

A morphism from $(X, \omega; \odot_{T,X})$ to $(Y, \omega; \odot_{T;Y})$ is a pair (f, u) where

1. $f : X \to Y$
2. $u : \omega \to \omega$
3. (f, u) preserves the truth of atomic formulas in the *forward* direction, which amounts to:

$$(e, x_1, x_2) \in (\odot_{T,X}) \implies (u(e), f(x_1), f(x_2)) \in (\odot_{T,Y}).$$

Such an $f : X \to Y$ is said to be uniformly order-preserving (**UOP**) and u is said to be a **uniformity function** for f.

Notation

$$e \odot_T x = \begin{cases} \varphi_e^x & \text{if } \varphi_e^x \in 2^\omega \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We write $e \odot_T x \simeq y$ instead of $(e, x, y) \in (\odot_T)$.

$\simeq$ denotes Kleene's equality.

Rephrasing

If $X, Y \subseteq 2^\omega$, $f : X \to Y$ is UOP if there exists $u : \omega \to \omega$ such that, for all $e \in \omega$ and $x_1, x_2 \in X$:

$$e \odot_T x_1 \simeq x_2 \implies u(e) \odot_T f(x_1) \simeq f(x_2),$$

or equivalently:

$$e \odot_T x_1 \text{ is defined and is in } X \implies u(e) \odot_T f(x_1) \simeq f(e \odot_T x_1).$$

For $x, y \in 2^\omega$, define ${}^s\odot_T$ by

$$(i,j)\ {}^s\odot_T\ x \simeq y \iff \begin{cases} i \odot_T x \simeq y \\ j \odot_T y \simeq x \end{cases}$$
$$\iff x \equiv_T y \textbf{ via } (i,j)$$

$$(i,j)\ {}^s\odot_T\ x = \begin{cases} \varphi_i^x & \text{if } \varphi_i^x \in 2^\omega \text{ and } \varphi_j^{\varphi_i^x} = x, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$f : X \to Y$ is uniformly Turing invariant (**UTI**) if there is $u : \omega^2 \to \omega^2$ such that

$$(i,j)\ {}^s\odot_T\ x_1 \simeq x_2 \implies u(i,j)\ {}^s\odot_T\ f(x_1) \simeq f(x_2),$$

or equivalently

$$(i,j)\ {}^s\odot_T\ x \text{ is defined and is in } X \implies u(i,j) \odot_T f(x) \simeq f\big((i,j)^s\odot_T x\big).$$

Note that:

If u witnesses that f is UOP, then $(i,j) \mapsto (u(i), u(j))$ witnesses that f is UTI.

Theorem 1

For all Turing degrees $\boldsymbol{x}$ and $\boldsymbol{y}$, the following are equivalent:

1. $\boldsymbol{x} \leq_T \boldsymbol{y}$
2. there is an embedding from $(\boldsymbol{x}, \omega; \odot_{T,\boldsymbol{x}})$ to $(\boldsymbol{y}, \omega; \odot_{T,\boldsymbol{y}})$
3. there exists a non-constant UTI function $f : \boldsymbol{x} \to \boldsymbol{y}$

Proof

1. $\implies$ 2. Take $y \in \boldsymbol{y}$ and define $f : \boldsymbol{x} \to \boldsymbol{y}, z \mapsto z \oplus y$. For $e \in \omega$, let $u(e)$ be the obvious natural number s.t.

$$\varphi_e^z = w \implies \varphi_{u(e)}^{z \oplus y} = w \oplus y.$$

(f, u) is the desired embedding.

2. $\implies$ 3. If (f, u) is one such embedding, then

- f is UOP, hence UTI
- f in injective, hence non-constant

3. $\implies$ 1. We need two lemmas.

Lemma 1

If

- $X \subseteq 2^\omega$ is closed under $\equiv_T$
- $f : X \to 2^\omega$ is UTI

then

- there is a *computable* uniformity function u for f.

Lemma 2

For all $y, z \in \omega^\omega$:

$$\bigoplus_n \varphi^y_{z(n)} \leq_T y \oplus z.$$

Proof.

Recall that $\left(\bigoplus_n \varphi^y_{z(n)}\right)(\langle i,j\rangle) = \varphi_{z(j)}(i)$.Take $U \in \omega$ such that $\varphi^y_i(j) = \varphi^y_U(\langle i,j\rangle)$ for all i, j, and you'll have

$$\left(\bigoplus_n \varphi^y_{z(n)}\right)(\langle i,j\rangle) = \varphi_{z(j)}(i) = \varphi^y_U(\langle i, z(j)\rangle) = \varphi^{y\oplus z}_{\tilde{U}}(\langle i,j\rangle). \quad \square$$

Remark

This is the crucial point that makes Theorem 1 hold for Turing case and fail for the arithmetic case.

Proof of 3. $\implies$ 1.

$f : \boldsymbol{x} \to \boldsymbol{y}$ is UTI and non-constant, so pick $z \equiv_T x$ such that

$$f(x) \neq f(z).$$

There is a computable function r such that

$$\varphi^x_{r(n)} = \begin{cases} x & \text{if } x(n) = 1, \\ z & \text{if } x(n) = 0. \end{cases}$$

There is $e \in \omega$ such that

$$\varphi^x_e = \varphi^z_e = x.$$

Setting $t : \omega \to \omega^2, n \mapsto (r(n), e)$, we get that t is computable and

$$t(n) \ {}^s\!\odot_T \ x \simeq \begin{cases} x & \text{if } x(n) = 1, \\ z & \text{if } x(n) = 0. \end{cases}$$

Thus

$$f\big(t(n)\,{}^{s}\!\odot_T x\big) \simeq \begin{cases} f(x) & \text{if } x(n) = 1, \\ f(z) & \text{if } x(n) = 0. \end{cases}$$

If $f(x)$ and $f(z)$ differ on the k-digit, then the k-th row of

$$\bigoplus_n f\big(t(n)\,{}^{s}\!\odot_T x\big)$$

is either x or $1 - x$. Hence:

$$\begin{aligned} x \leq_T \bigoplus_n f\big(t(n)\,{}^{s}\!\odot_T x\big) &= \bigoplus_n \big(u(t(n))\,{}^{s}\!\odot_T f(x)\big) \\ &= \bigoplus_n \varphi^{f(x)}_{\pi(u(t(n)))} \leq_T f(x) \oplus (\pi \circ u \circ t) \leq_T f(x) \end{aligned}$$

where u is a computable uniformity function and $\pi : \omega^2 \to \omega$ is the projection on the first coordinate.

□

We proved:

Theorem

If $f : [x]_{\equiv_T} \to 2^\omega$ is UTI, then

- either $f(x) \geq_T x$
- or f is constant

Compare this with:

Uniform Martin's conjecture, part I (proved by Slaman and Steel, 1988)

Assume AD. If C is a cone and $f : C \to 2^\omega$ is UTI, then

- either $f(x) \geq_T x$ on a cone
- or f is constant on a cone (up to $\equiv_T$).

Corollary

Assume TD. If C is a cone and $f : C \to 2^\omega$ is UTI, then

- either $f(x) \geq_T x$ on a cone
- or f is (literally) constant on a cone

Proof.

$$A = \{\, x \in C \mid f \upharpoonright [x]_{\equiv_T} \text{ is constant} \,\}$$

is Turing invariant, so by TD either A or $2^\omega \setminus A$ contains a cone.
If $2^\omega \setminus A$ contains a cone D (wlog: $D \subseteq C$), then for all $x \in D$
$f \upharpoonright [x]_{\equiv_T}$ is a non-constant UTI function, and so $f(x) \geq_T x$.
If A contains a cone, then next Lemma applies. □

Lemma

Assume TD. Suppose that $f : 2^\omega \to 2^\omega$ satisfies

$$x \equiv_T y \implies f(x) = f(y)$$

for all $x, y \geq_T z$. Then f is (literally) constant on a cone.

Theorem

The following are equivalent over ZF+DC:

1. TD
2. if f is UTI on a cone, then either $f(x) \geq_T x$ on a cone, or f is literally constant on a cone
3. if f is UTI on a cone, then either $f(x) \geq_T x$ on a cone, or f is constant up to $\equiv_T$ on a cone

Proof.

1. $\implies$ 2. is the previous Corollary. 2. $\implies$ 3. is trivial.
3. $\implies$ 1.: if A is Turing invariant, define

$$f(x) = \begin{cases} \underline{0} = 000\ldots & \text{if } x \in A, \\ \underline{0}' & \text{if } x \in 2^\omega \setminus A. \end{cases}$$

f is UTI and it cannot be that $f(x) \geq_T x$ on a cone. So either $f(x) \equiv_T \underline{0}$ on a cone, or $f(x) \equiv_T \underline{0}'$. In the former case, A contains a cone, in the latter $2^\omega \setminus A$ does. □

What about part II?

Define

- $f \leq_M g \iff f(x) \leq_T g(x)$ on a cone
- $f'(x) = \big(f(x)\big)'$
- f is increasing on $A \iff f(x) \geq_T x$, for all $x \in A$.

Uniform Martin's conjecture, part II (proved by Steel, 1983)

Assume AD. The set

$$J = \{\, f : 2^\omega \to 2^\omega \mid f \text{ is both UTI and increasing on a cone} \,\}$$

is pre-well-ordered by $\leq_M$. Moreover, there are no $f, g \in J$ such that $f <_M g <_M f'$.

Question

Does this theorem arise locally?

- That is: is it derivable from some property that UTI functions exhibit locally?
- and also: is it provable from TD?

Theorem (Chong-Wang-Yu, 2010)

Part II of projective uniform Martin's conjecture is equivalent to PD over ZFC.

Hence, if part II of projective uniform Martin's conjecture were provable from projective TD, we would have

$$\text{projective TD} \implies \text{PD}.$$

Pointclass jump operators

Given a reasonable (ω-parametrized, contains all computable sets, and has the substitution property) lightface pointclass Γ, we say $f :\subseteq 2^\omega \to 2^\omega$ is a Γ-jump operator if

$$f(x) \equiv_T \text{the universal } \Gamma(x) \text{ subset of } \omega.$$

Example

- $x \mapsto x^{(\alpha)}$ is a Σ^0_α-jump operator, for $\alpha < \omega_1$
- $x \mapsto \mathcal{O}^x$ is a Π^1_1-jump operator

Theorem (Becker, 1988)

Assuming AD, if f is UTI and increasing on a cone, then

- either $f(x) \equiv_T x$ on a cone
- or f is a Γ-jump operator on a cone, for some reasonable Γ

One deduces part II of uniform Martin's conjecture from this, using that reasonable pointclasses are well-ordered.

Becker's theorem, locally?

Fix a Turing degree $\boldsymbol{x}$, and consider the smallest family $\mathcal{U}_{\boldsymbol{x}}$ of functions $f : \boldsymbol{x} \to 2^\omega$ that contains

- constant functions
- $x \mapsto x$ and $x \mapsto \bar{x}$
- pointclass jump operators

and is closed under

- precomposition with r, for every computable isomorphism r
- (finite and) infinite joins.

Every function in $\mathcal{U}_{\boldsymbol{x}}$ is UTI (indeed UOP).

Question

Is every UTI $f : \boldsymbol{x} \to 2^\omega$ in $\mathcal{U}_{\boldsymbol{x}}$?

The intent of Martin's conjecture, part II was to assert thoroughly that "natural" Turing degrees are well-ordered by $\leq_T$.

Idea

- A "natural" degree can be relativized, leading to an easy "definable" increasing TI function.
- problem: you could patch together several definable functions into a single definable function... what should such a collage represent?
- Under AD, one piece has to prevail on a cone, so $\equiv_M$-classes of increasing TI functions should capture the "fundamental" definable functions.

The underlying question here is: are pointclass jump operators the only *fundamental* ways to increase the computational content of Turing degrees?

Question

Is every UTI $f : \boldsymbol{x} \to 2^\omega$ in $\mathcal{U}_{\boldsymbol{x}}$?

Thank you!

Lemma

If

- $X \subseteq 2^\omega$ is closed under $\equiv_T$
- $f : X \to 2^\omega$ is UOP

then

- there is a *computable* uniformity function u for f.

Proof.

Consider the (obvious) binary operation $*_T$ on ω that leads

$$\varphi_i^{\varphi_j^x} = \varphi_{i *_T j}^x.$$

Call $*_T$ **Turing multiplication**. Note that $*_T$ is computable. $(\omega, *_T)$ is a semi-group and

- $j \odot_T x$ is defined $\implies$ $i \odot_T (j \odot_T x) \simeq (i *_T j) \odot_T x$.

Suppose f is UOP, so there is v s.t.

$$e \odot_T x \simeq y \implies v(e) \odot_T f(x) \simeq f(y).$$

Let $a, b, c \in \omega$ s.t.

$$\varphi_c^x = 1^\frown x \qquad \varphi_b^x = 0^\frown x \qquad \varphi_a^{0^e 1^\frown x} = \varphi_e^x.$$

Let $ab^e c$ be shorthand for $a *_T \underbrace{b *_T \cdots *_T b}_{e} *_T c$.

Note that $e \odot_T x \simeq (ab^e c) \odot_T x$.
Now, if $e \odot_T x$ is defined and is in X:

$$\begin{aligned} f(e \odot_T x) \simeq f\left((ab^e c) \odot_T x\right) &\simeq v(a) \odot_T f\left((b^e c) \odot_T x\right) \\ &\vdots \\ &\simeq v(a)v(b)^e v(c) \odot_T f(x) \end{aligned}$$

So define $u(e) = v(a)v(b)^e v(c)$and you'll have

- $f(e \odot_T x) \simeq u(e) \odot_T f(x)$, whenever $e \odot_T x$ is defined and is in X
- u is computable

□

If f is UTI and v is a uniformity function for it, let:

$$\varphi_c^x = 1^\frown x \qquad \varphi_b^x = 0^\frown x \qquad \varphi_m^x(n) = x(n+1).$$

We have $(c, m)\,{}^s\!\odot_T x \simeq 1^\frown x$ and $(b, m)\,{}^s\!\odot_T x \simeq 0^\frown x$. Let $d \in \omega$ be such that

$$\varphi_d^{0^i 1 0^j 1^\frown x} = 0^j 1 0^i 1^\frown \varphi_i^x.$$

We have

$$\begin{aligned}
(i, j)\,{}^s\!\odot_T x \simeq y \iff (d, d)\,{}^s\!\odot_T 0^i 1 0^j 1^\frown x \simeq 0^j 1 0^i 1^\frown y \\
\iff (d, d)(b, m)^i (c, m)(b; m)^j (c, m)\,{}^s\!\odot_T x \\
\simeq (b, m)^j (c, m)(b, m)^i (c, m)\,{}^s\!\odot_T y \\
\implies (m, c)(m, b)^i (m, c)(m, b)^j (d, d)(b, m)^i (c, m)(b; m)^j (c, m)\,{}^s\!\odot_T x \\
\simeq y.
\end{aligned}$$

So define $u(i, j)$ as

$$v(m, c) v(m, b)^i v(m, c) v(m, b)^j v(d, d) v(b, m)^i v(c, m) v(b; m)^j v(c, m).$$