A new basis theorem for Σ^1_3

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Π^1_2-singletons and $0^\#$

Theorem (Jensen)

Suppose $x \in \mathbb{R}$ and $x \in L[0^\#]$. Then one of the following hold.

(1) $x^\# \in L[0^\#]$.

(2) $0^\# \in L[x]$.

Theorem (Kechris-Harrington)

Suppose for all $x \in \mathbb{R}$, $x^\#$ exists, and that y is a Π^1_2-singleton. Then one of the following hold.

(1) $y \in L[0^\#]$.

(2) $0^\# \in L[y]$.

Theorem (Kechris-Harrington)

Suppose for all $x \in \mathbb{R}$, $x^\#$ exists, and that y is a Π^1_2-singleton. Then:

- *Either $y^\# \in L[0^\#]$ or $0^\# \in L[y]$.*

Π^1_2-singletons and Solovay's conjecture

Conjecture (Solovay)

Suppose $0^\#$ exists. Then there exists y such that:

- y is a Π^1_2-singleton.
- $y \notin L$ and $0^\# \notin L[y]$.
  - $y \notin L$ and $y^\# \in L[0^\#]$.

Theorem (S. Friedman)

Suppose $0^\#$ exists. Then there is a Π^1_2-singleton y such that

- $y \notin L$ *and* $0^\# \notin L[y]$.

- Friedman's singleton is very close to L (same cardinals).
- It is not known if Friedman singletons can be required to be Π^1_2-singletons in **all** transitive $M \models \mathrm{ZC}$ to which they belong.

Generalizing $0^{\#}$: α-quasi-$\#$ singletons

Definition

Suppose $x \in \mathbb{R}$ is and $\alpha < \omega_1$. Then x is an α-**quasi**-$\#$ if for some Π^1_2-formula $\varphi(t_0)$:

- $V \models \varphi[x]$.
- Suppose $y \in \mathbb{R}$, $\beta < \omega_1$, $L_\beta[x, y] \models \varphi[y]$, and
  - $L_\beta[x, y] \models \mathrm{ZFC} \backslash \mathrm{Powerset}$.

  Then $x = y$.
- Suppose $z \in \mathbb{R}$ codes S, S is a set of x-admissible ordinals, and that S has ordertype α.
  - Then x is hyper-arithmetic in z.

Suppose x is an ω-quasi-$\#$ and $x \notin L$. Then $L[x]$ cannot be close to L:

- L and $L[x]$ cannot have the same cardinals.
  - L cannot contain any infinite set of $L[x]$ cardinals.

The Strong Solovay Conjecture

Lemma

- $0^\#$ *is an* ω*-quasi-*$\#$
  - *Every* ω*-quasi-*$\#$ *is recursive in* $0^\dagger$*.*
- $0^\dagger$ *is an* ω^2*-quasi-*$\#$ *and* $0^\dagger$ *is not an* ω*-quasi-*$\#$*.*

Question

- Suppose $0^\#$ exists, x is an α-quasi-$\#$ for some $\alpha < \omega_1$, and $x \notin L$. Must $0^\# \in L[x]$?
- **Strong Solovay Conjecture:** No

Relativising to x: (α, x)-**quasi**-$\#$ singletons

Definition

Suppose $x, y \in \mathbb{R}$ and $\alpha < \omega_1$. Then y is an (α, x)-**quasi**-$\#$ if for some Π^1_2-formula $\varphi(t_0, t_1)$:

- $V \models \varphi[x, y]$.
- Suppose $z \in \mathbb{R}$, $\beta < \omega_1$, and:
  - $L_\beta[x, y, z] \models \varphi[x, z]$,
  - $L_\beta[x, y, z] \models \mathrm{ZFC} \backslash \mathrm{Powerset}$.

  Then $z = y$.
- Suppose $z \in \mathbb{R}$ codes (S, x), S is a set of (x, y)-admissible ordinals, and that S has ordertype α.
  - Then y is hyper-arithmetic in z.

The Strong Solovay Conjecture holds on a cone

Theorem (PD)

There exists $\alpha < \omega_1$ such that for a Turing cone of x:

- *There exists y such that*
  - *y is an (α, x)-quasi-$\#$ and $x \leq_T y$.*
  - *$y \notin L[x]$ and $y \in L[x^\#]$*
  - *$x^\# =_T y^\#$.*

- Let $\mathbb{P}_x$ be the partial order of all y such that y is an (α, x)-quasi-$\#$ for some α, ordered by $L[x]$-reducibility.
  - What is the theory of $\mathbb{P}_x$ on a cone?

The effective structure of the projective sets

Theorem (Moschovakis)

Assume PD *and* $0 < n < \omega$. *Then*

- *If* n *is even then there is a largest countable* Σ^1_n*-set*
  - *This is denoted* C_n.
- *If* n *is odd then the* Π^1_n*-singletons are prewellordered under* Δ^1_n*-reducibility.*

Theorem (Kechris)

Assume PD, $0 < n < \omega$, *and* n *is odd.*

- *Then there is a largest countable* Π^1_n*-set*
  - *This is denoted* C_n.

- $C_2 = \mathbb{R} \cap L$.
- Every element of C_2 is recursive in some element of C_1.

The classical theory

Definition

Q_1 is the set of all $x \in \mathbb{R}$ such that $x \leq_T y$ for some y such that y is Δ^1_1-definable.

- $Q_1 = \mathbb{R} \cap L_\eta$ where η is the least admissible ordinal.
  - $Q_1 = \mathrm{HYP}$.

Theorem (Spector-Gandy)

Let η be the least admissible ordinal. Suppose that $X \subseteq Q_1$. Then the following are equivalent.

(1) *X is Π^1_1.*

(2) *X is Σ^1_1 in L_η.*

The Σ^1_1-basis theorems

Theorem (Harrison)

Suppose X is a countable Σ^1_1-set. Then X contains only Δ^1_1-reals.

Theorem (Moschovakis)

Suppose that x is a Π^1_1-singleton. Then one of the following hold.

(1) *There exists $y \subset \omega$ such that $x \leq_T y$ and such that y is Δ^1_1-definable.*

(2) *Every nonempty Σ^1_1 set contains an element which is $\Delta^1_1(x)$.*

Theorem (Kleene)

Kleene's $\mathcal{O}$ is the least Π^1_1-singleton under Δ^1_1-reducibility which is not in Q_1.

Question

How does this generalize to Σ^1_{n+1} where $n > 0$ and $n + 1$ is odd?

The Cabal Dawn: Kechris-Martin-Solovay's Q-theory

Theorem (Martin-Solovay)

Assume PD, $0 < n < \omega$, and $n+1$ is odd. Suppose x is a Π^1_{n+1}-singleton. Then one of the following hold.

(1) *There exists $y \subset \omega$ such that $x \leq_T y$ and such that y is $\Delta^1_{n+1}(\alpha)$-definable for some $\alpha < \omega_1$:*
   - *i.e. y is Δ_n-definable from α in $H(\omega_1)$.*

(2) *Every nonempty Σ^1_{n+1} set contains an element which is $\Delta^1_{n+1}(x)$.*

- Q_{n+1} denotes the set of all $x \in \mathbb{R}$ such that there exists $y \subset \omega$ such that:
  - $x \leq_T y$
  - y is $\Delta^1_{n+1}(\alpha)$-definable for some $\alpha < \omega_1$.

The uniform projective picture

Assume PD and suppose $0 < n < \omega$. Let X be the set of all $z \in \mathbb{R}$ such that there exists $y \subset \omega$ such that:

- $z \leq_T y$
- y is $\Delta^1_n(\alpha)$-definable for some $\alpha < \omega_1$.

- *Suppose n is* **even**. *Then $X = C_n$*
  - *where C_n is the largest countable Σ^1_n-set.*
- *Suppose n is* **odd**. *Then $X = Q_n$, Q_n is a Π^1_n-set, and $Q_n \subseteq C_n$*
  - *where C_n is the largest countable Π^1_n-set.*

Mitchell–Steel models

Definition

For each $0 < n < \omega$, M_n is the Mitchell-Steel least proper class iterable Mitchell-Steel model with n Woodin cardinals.

- Suppose M_n exists for all $n < \omega$. Then
  - $M_n^\#$ exists for all $n < \omega$,
  - $M_n^\#$ is characterized by just being α-iterable for all $\alpha < \omega_1$.

Theorem (Cabal)

The following are equivalent.

(1) PD.

(2) *For each $n < \omega$, $M_n^\#$-exists.*

A unified theory: Fine-structure meets determinacy I

Theorem (Cabal)

Assume PD, $0 < n < \omega$, *and* n *is* **even**. *Then:*

- $M_n^\#$ *exists and* $M_n^\#$ *is a* Π^1_{n+2}*-singleton.*
- $\mathbb{R} \cap M_n = C_{n+2}$.

- $M_n^\#$ is the generalization of Silver's $0^\#$ to level $n+2$.

Theorem (Cabal)

Assume PD, $0 < n < \omega$, *and* n *is* **odd**. *Then:*

- $M_n^\#$ *exists and* $M_n^\#$ *is a* Π^1_{n+2}*-singleton.*
- $\mathbb{R} \cap M_n = Q_{n+2}$.
- $M_n^\#$ *is the least* Π^1_{n+2}*-singleton under* Δ^1_{n+2}*-reducibility which is not in* Q_{n+2}.

- $M_n^\#$ is the generalization of Kleene's $\mathcal{O}$ to level $n+2$.

A unified theory: Fine-structure meets determinacy II
The basis theorem at odd levels

Theorem

Assume PD, $0 < n < \omega$, and n is odd. Let δ be the least Woodin cardinal of M_n and suppose X is a nonempty Σ^1_{n+2} set.

- *Then there exists a partial order $\mathbb{P}$ which is δ-cc in M_n and a term $\tau \in M_n$ such that* **every** *M_n-generic filter on $\mathbb{P}$ interprets τ as a member of X.*

- This is the best possible basis theorem.
  - It implies X contains a member which is recursive in $M_n^\#$ and much more.

What about Π^1_{n+2}-singletons and $M_n^\#$ when n is even?

- (PD) Suppose that $0 < n < \omega$ and n is even.
  - $C_{n+2}(x)$ is the largest countable $\Sigma^1_{n+2}(x)$-set.
  - For each $x \in \mathbb{R}$, $M_n(x)$ denotes M_n relatived to x.
    - $M_n^\#(x)$ is a $\Pi^1_{n+2}(x)$-singleton.
    - $\mathbb{R} \cap M_n(x) = C_{n+2}(x)$

Theorem (PD)

Suppose that $0 < n < \omega$ and n is even.

- *Then for a Turing cone of x there exists y such that:*
  - *y is a $\Pi^1_{n+2}(x)$-singleton and $y \leq_T M_n^\#(x)$.*
  - *$y \notin C_{n+2}(x)$ and $M_n^\#(x) \notin C_{n+2}(y)$.*
  - *$M_n^\#(y) =_T M_n^\#(x)$.*

Just beyond projective: Π^1_ω and $\Sigma^1_{\omega+1}$

- Π^1_ω *is analogous to* Π^0_1 *(with* $\mathbb{R} = \omega^\omega$*).*
- $\Sigma^1_{\omega+1}$ *is analogous to* Σ^1_1*.*

Suppose $A \subset \mathbb{R}$. Then A is $\Sigma^1_{\omega+1}$-definable if for some formula $\varphi(x_0)$ and for all $z \in \mathbb{R}$:

- $z \in A$ if and only if there exists a transitive set M such that
  - $M \models \mathrm{ZC}$,
  - (projective correctness) $M \prec_{\Sigma^1_n} V$ for all $n < \omega$,
  - $z \in M$ and $M \models \varphi[z]$.

Q-theory at $\Sigma^1_{\omega+1}$

Theorem (Moschovakis)

Assume AD. *Then the* $\Pi^1_{\omega+1}$*-singletons are wellordered under* $\Delta^1_{\omega+1}$*-reducibility.*

Theorem (Kechris-Martin-Solovay)

Assume AD. *Suppose* x *is a* $\Pi^1_{\omega+1}$*-singleton. Then one of the following hold.*

(1) *There exists* $y \subset \omega$ *such that* $x \leq_T y$ *and such that* y *is a* $\Pi^1_\omega(\alpha)$*-singleton for some* $\alpha < \omega_1$

(2) *Every nonempty* $\Sigma^1_{\omega+1}$ *set contains an element which is recursive in some* $\Pi^1_\omega(x)$*-singleton.*

- $Q_{\Pi^1_{\omega+1}}$ denotes the set of all $x \in \mathbb{R}$ such that there exists $y \subset \omega$ such that:
  - $x \leq_T y$ and y is $\Pi^1_\omega(\alpha)$-singleton for some $\alpha < \omega_1$.

The $Q_{\Pi^1_{\omega+1}}$ questions

Theorem (Cabal)

Assume AD.

- *Suppose* $x \in \mathbb{R}$. *Then the following are equivalent.*
  (1) $x \in M_1$.
  (2) $x \in Q_3$.
  (3) *There exists* $y \in \mathbb{R}$ *such that* y *is a* $\Pi^1_2(\alpha)$*-singleton for some* $\alpha < \omega_1$ *and such that* $x \leq_T y$.
- $M_1^{\#}$ *is the least* Π^1_3*-singleton under* Δ^1_3*-reducibility which is not in* Q_3.

Questions

- What is the analogous characterization of $Q_{\Pi^1_{\omega+1}}$?
  - For example, what plays the role of M_1?
- What is the least $\Pi^1_{\omega+1}$-singleton which is not in $Q_{\Pi^1_{\omega+1}}$?
  - For example, what plays the role of $M_1^{\#}$?

n-full Mitchell-Steel models

- Suppose $b \subset \alpha$ for some $\alpha < \omega_1$. Then:
  - $M_n^\#(b)$ is $M_n^\#$ relatived to b.

Definition

Assume AD and that M is a countable iterable Mitchell-Steel model such that

$$M \models \mathrm{ZC}.$$

Suppose $0 < n < \omega$. Then M is ***n*-full** if for all $\gamma < \mathrm{Ord}^M$:

$$M_n^\#(M|\gamma)) \in M.$$

n-full models and generic projective correctness

Theorem (Cabal)

Assume AD and that M is a countable iterable Mitchell-Steel model such that

$$M \models \mathrm{ZC}.$$

Then the following are equivalent.

(1) *M is n-full for all $n < \omega$.*

(2) *Suppose g is M-generic. Then*

$$M[g] \cap V_{\omega+1} \prec V_{\omega+1}.$$

Rudominer Stacks

Definition

Assume AD. A countable iterable (sound) Mitchell-Steel model $M \models \mathrm{ZC}$ is a **Rudominer Stack** if:

- M is n-full for all $n < \omega$.
- For each $n < \omega$, let γ_n be the least γ such that
  - γ is a cardinal of M.
  - γ is a Woodin cardinal in $M_n(M|\gamma)$.

  Then γ_n exists for all $n < \omega$ and $\mathrm{Ord}^M = \sup\{\gamma_n \mid n < \omega\}$.

Definition

Assume AD. A countable iterable Mitchell-Steel model $M \models \mathrm{ZC}$ is an **Admissible Rudominer Stack** if:

- M is a Rudominer Stack.
- $M = V_{\mathrm{Ord}^M} \cap L_\eta(M)$ where $\eta > \mathrm{Ord}^M$ is the least ordinal which is admissible relative to M.

Rudominer's question

Theorem (Rudominer)

Assume AD. *Let* M_0 *be the Mitchell-Steel least Rudominer Stack.*

- *Then for each* $x \in M_0 \cap \mathbb{R}$ *there exists* $y \in M_0 \cap \mathbb{R}$ *such that*
  - $x \leq_T y$.
  - y *is a* $\Pi^1_\omega(\alpha)$*-singleton for some* $\alpha < \omega_1$.

Theorem (Rudominer)

Assume AD. *Let* M_0^+ *be the Mitchell-Steel least Admissible Rudominer Stack.*

- *Suppose* y *is a* $\Pi^1_\omega(\alpha)$*-singleton for some* $\alpha < \omega_1$.

Then $y \in M_0^+$.

Question (Rudominer)

What is the Mitchell-Steel least M (projecting to ω) such that M contains all y such that y is a $\Pi^1_\omega(\alpha)$-singleton for some $\alpha < \omega_1$?

The characterization of reals recursive in $\Pi^1_\omega(\alpha)$-singletons

Theorem (AD)

Let M_0 be the Mitchell-Steel least Rudominer Stack.

- *Suppose y is a $\Pi^1_\omega(\alpha)$-singleton for some $\alpha < \omega_1$.*

Then $y \in M_0$.

Theorem (AD)

Let M_0 be the Mitchell-Steel least Rudominer Stack and let

$$\sigma = Q_{\Pi^1_{\omega+1}}.$$

Then

- $\sigma = \mathbb{R} \cap M_0$.
- $\sigma = \mathbb{R} \cap L(\sigma)$.

Completing the fine-structure analysis of Q-theory at $\Sigma^1_{\omega+1}$

Theorem (AD)

Let M_0 be the Mitchell-Steel least Rudominer Stack and let T_{M_0} be the theory of M_0. Then

- $\mathbb{R} \cap M_0 = Q_{\Pi^1_{\omega+1}}$.
- *T_{M_0} is a $\Pi^1_{\omega+1}$-singleton.*
- *T_{M_0} is the least $\Pi^1_{\omega+1}$-singleton under $\Delta^1_{\omega+1}$-reducibility which is not in $Q_{\Pi^1_{\omega+1}}$.*
- *Suppose $X \subseteq \mathbb{R}$ is $\Sigma^1_{\omega+1}$ and $X \neq \emptyset$.*
  - *Then X contains an element which is recursive in T_{M_0}.*

The Kechris-Martin Theorem for $\Pi^1_{\omega+1}$

Notation (PD):

$$T^1_\omega = \bigoplus T_\varphi$$

where φ ranges over all Σ^1_n-formulas such that $0 < n < \omega$ and φ defines a Σ^1_n-scale on a Σ^1_n-set, and T_φ is the tree of that scale.

Theorem (AD)

Let η be the least admissible ordinal relative to T^1_ω. Then $\mathbb{R} \cap L_\eta[T^1_\omega] = Q_{\Pi^1_{\omega+1}}$ and $\mathbb{R} \cap L_\eta[T^1_\omega] \in L_\eta[T^1_\omega]$.

Theorem (AD)

Let η be the least admissible ordinal relative to T^1_ω. Suppose $X \subseteq Q_{\Pi^1_{\omega+1}}$. Then the following are equivalent.

(1) *X is $\Pi^1_{\omega+1}$.*

(2) *X is $\Sigma^1_{\omega+1}$-definable in $L_\eta[T^1_\omega]$.*

(3) *X is $\Sigma_1(T^1_\omega)$-definable in $L_\eta[T^1_\omega]$.*

Back to Σ^1_3

Theorem (Moschovakis)

Assume PD *and that* $X \subset \mathbb{R}$ *is a countable* Σ^1_3*-set.*

- *Then there is a* Π^1_2*-singleton* y *such that every element of* X *is recursive in* y*.*

Definition

Suppose E is an equivalence relation on some set $Z \subset \mathbb{R}$.

- A set $X \subseteq Z$ is E-**invariant** if X is a union of E-equivalence classes.

- The E-**closure** of a set $X \subseteq Z$ is the union of all $[a]_E$ such that $a \in X$.

The E-basis theorem for Σ^1_3

Theorem (PD)

Suppose that E is a Σ^1_2-equivalence relation on some Σ^1_2 set Y. Suppose that $X \subseteq Y$ is an E-invariant set such that X is a Σ^1_3-set. Suppose X contains only countably many E-equivalence classes.

- *Then there is a Π^1_2-singleton y such that every E-equivalence class of X contains an element which is recursive in y.*

- The restriction to Σ^1_2-equivalence relations is best possible.
  - There is a nonempty Π^1_2-set with no member in Q_3.

Connections with a question of Kechris

Lemma (Kechris)

Suppose that E is a Σ^1_2-equivalence relation on some Σ^1_2 set Y. Suppose that $X \subseteq Y$ is an E-invariant set such that X is a Σ^1_3-set. Suppose X contains just **one** *E-equivalence class.*

- *Then X is Δ^1_3.*

Question (Kechris)

(PD) Suppose that X is a Δ^1_3-set and that X is $\underset{\sim}{\Sigma}^1_2$. Must X be $\Sigma^1_2(t)$ for some Δ^1_3-real t?

- If yes then X must contain a Δ^1_3-real

Theorem (Hjorth)

(PD) *Suppose that X is a Δ^1_3-set and that X is $\underset{\sim}{\Sigma}^1_2$.*

- *Then X is $\Sigma^1_2(t)$ for some Δ^1_3-real t.*

What about $\Sigma^1_{\omega+1}$?

Theorem (Moschovakis)

Assume $\mathrm{AD}^{L(\mathbb{R})}$ and that $X \subset \mathbb{R}$ is a countable $\Sigma^1_{\omega+1}$-set.

- *Then there is a Π^1_ω-singleton y such that every member of X is recursive in y.*

- Is there a fine-structure proof?

Question (E-basis problem for $\Sigma^1_{\omega+1}$)

($\mathrm{AD}^{L(\mathbb{R})}$) Fix $n < \omega$. Suppose that E is a Σ^1_n-equivalence relation on $\mathbb{R}$ and that $X \subseteq \mathbb{R}$ is an E-invariant set such that X is a $\Sigma^1_{\omega+1}$-set. Suppose X contains only countably many E-equivalence classes.

- Must there be a Π^1_ω-singleton y such that every E-equivalence class of X contain an element recursive in y?

- This problem is related to one about E-invariant measures.

E-invariant measures

Let σ_{PROJ} be the smallest σ-algebra containing all the projective sets.

- Suppose E is a projective equivalence relation on $\mathbb{R}$.

Let $\sigma_{\mathrm{PROJ}}(E)$ denote all the E-invariant sets in σ_{PROJ}.

Definition

Suppose μ is an ultrafilter on $\sigma_{\mathrm{PROJ}}(E)$.

- μ is **indecomposable** if for any prewellordering $\leq_P \in \sigma_{\mathrm{PROJ}}$ on $\mathbb{R}$ with E-invariant classes, there exists $x \in \mathbb{R}$ such that
$$[x]_P \in \mu$$
where $[x]_P = \{y \in \mathbb{R} \mid x \leq_P y \text{ and } y \leq_P x\}$.
- μ is **borel generated** if μ is generated by the E-closures of borel sets.

Notation:

- $\mathfrak{U}(\sigma_{\mathrm{PROJ}}(E))$ denotes the set of all ultrafilters on $\sigma_{\mathrm{PROJ}}(E)$ which are indecomposable and borel generated.

A classification problem

Question

Suppose there is a proper class of Woodin cardinals, E is a projective equivalence relation, and that

$$\mu \in \mathfrak{U}(\sigma_{\mathrm{PROJ}}(E)).$$

- Must μ be universally Baire?
  - CH: Probably not.
- Suppose μ is universally Baire. Must μ be projective?
  - This is related to the E-basis problem for $\Sigma^1_{\omega+1}$.

Theorem (Iterable Model Hypothesis)

Suppose there is a proper class of measurable Woodin cardinals and that CH holds.

- *Suppose E is a projective equivalence relation.*

Then the following are equivalent.

(1) *There exists $\mu \in \mathfrak{U}(\sigma_{\mathrm{PROJ}}(E))$ which is* **not** *universally Baire.*

(2) $|\mathfrak{U}(\sigma_{\mathrm{PROJ}}(E))| = 2^{\mathfrak{c}}$.