Counting Woodin cardinals in HOD

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AD^+

- (ZF) Θ is the supremum of the ordinals α such that there exists a surjection, $\pi : \mathbb{R} \to \alpha$.

Definition: AD^+ Axioms

1. $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$
2. Suppose $\pi : \gamma^\omega \to \omega^\omega$ is continuous and $\gamma < \Theta$.
    - Then for each $A \subseteq \omega^\omega$, $\pi^{-1}(A)$ is determined.
3. (Suppose $A \subset \omega^\omega$ then $A \in L(S, \mathbb{R})$ for some $S \subseteq \mathrm{Ord}$.)

By the Moschovakis Coding Lemma:

Theorem (AD^+)

Suppose $\Gamma \subset \mathcal{P}(\mathbb{R})$*.*

- *Then* $L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$*.*

The utility of AD^+: Part I

- Many consequences of $\mathrm{AD}^{L(\mathbb{R})}$ for $L(\mathbb{R})$ generalize to $L(\mathcal{P}(\mathbb{R}))$ assuming AD^+.

Definition (AD)

$M_{\underset{\sim}{\Delta}^2_1}$ is the union of all transitive sets M such that

$$(M, \in) \cong (\mathbb{R}/\sim, E)$$

for some $E, \sim$ which are each $\underset{\sim}{\Delta}^2_1$-definable binary relations.

- Similarly define $M_{\Delta^2_1}$.

Theorem (AD^+)

The following hold.

- Σ^2_1 *has the scale property.*
- *Suppose* $X \subset L(\mathcal{P}(\mathbb{R}))$ *is* $\Sigma_1(\mathbb{R})$*-definable in* $L(\mathcal{P}(\mathbb{R}))$*.*
  - *If* $X \neq \emptyset$ *then* $X \cap M_{\Delta^2_1} \neq \emptyset$*.*
- $M_{\underset{\sim}{\Delta}^2_1} \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$*.*

Lecture 1: The Kechris-Martin Theorem for $\Pi^1_{\omega+1}$

Notation (PD):

$$T^1_\omega = \bigoplus T_\varphi$$

where φ ranges over all Σ^1_n-formulas such that $0 < n < \omega$ and φ defines a Σ^1_n-scale on a Σ^1_n-set, and T_φ is the tree of that scale.

Theorem ($\mathrm{AD}^{L(\mathbb{R})}$)

Let η be the least admissible ordinal relative to T^1_ω. Then

- $\mathbb{R} \cap L_\eta[T^1_\omega] = Q_{\Pi^1_{\omega+1}}$ *and* $\mathbb{R} \cap L_\eta[T^1_\omega] \in L_\eta[T^1_\omega]$.

Theorem ($\mathrm{AD}^{L(\mathbb{R})}$)

Let η be the least admissible ordinal relative to T^1_ω. Suppose $X \subseteq Q_{\Pi^1_{\omega+1}}$. Then the following are equivalent.

(1) X *is* $\Pi^1_{\omega+1}$.

(2) X *is* $\Sigma^1_{\omega+1}$*-definable in* $L_\eta[T^1_\omega]$.

(3) X *is* $\Sigma_1(T^1_\omega)$*-definable in* $L_\eta[T^1_\omega]$.

The utility of AD^+: Part II

Definition

Suppose $A \subseteq \mathbb{R}$. Then A is **strongly projective-like** if the pointclass, $\Sigma^1_{n+1}(A)$, has the scale property for some $n < \omega$.

- $T^1_\omega(A) = \bigoplus T_\varphi$ is defined exactly as T^1_ω is defined
  - using $\Sigma^1_n(A)$-formulas in place of Σ^1_n-formulas.

Definition (AD^+)

Suppose that $A \subset \mathbb{R}$ is strongly projective-like. Then

- $Q_{\Pi^1_{\omega+1}(A)}$ is the set of all x such that there exists y such that:
  - $x \leq_T y$ and y is a $\Pi^1_\omega(A)(\alpha)$-singleton for some $\alpha < \omega_1$.

Theorem (AD^+)

Suppose that $A \subset \mathbb{R}$ is strongly projective-like. Let η be the least admissible ordinal relative to $T^1_\omega(A)$. Then

- $\mathbb{R} \cap L_\eta[T^1_\omega(A)] = Q_{\Pi^1_{\omega+1}(A)}$ *and* $\mathbb{R} \cap L_\eta[T^1_\omega(A)] \in L_\eta[T^1_\omega(A)]$.

The Kechris-Martin Theorem for $\Pi^1_{\omega+1}(A)$

- $T^1_\omega(A)$ is a subtree of $(\omega \times \delta^1_\omega(A))^{<\omega}$

where $\delta^1_\omega(A)$ is the supremum of the lengths of all the prewellorderings which are $\Sigma^1_n(A)$ for some $n < \omega$.

Theorem (AD^+)

Suppose that $A \subset \mathbb{R}$ is strongly projective-like. Let η be the least admissible ordinal relative to $T^1_\omega(A)$. Suppose

- $X \subseteq Q_{\Pi^1_{\omega+1}(A)}$.

Then the following are equivalent.

(1) *X is $\Pi^1_{\omega+1}(A)$.*

(2) *X is $\Sigma^1_{\omega+1}(A)$-definable in $L_\eta[T^1_\omega(A)]$.*

(3) *X is $\Sigma_1(T^1_\omega(A))$-definable in $L_\eta[T^1_\omega(A)]$.*

- The analysis shows that for all $\alpha < \delta^1_\omega(A)$:
  - $\mathcal{P}(\alpha) \cap L_\eta[T^1_\omega(A)] \in L_\eta[T^1_\omega(A)]$.

Basic notions: Suslin sets and Suslin cardinals

Definition (ZF)

A set $A \subseteq \mathbb{R}$ is λ-**Suslin** if A has a λ-scale.

Definition (ZF)

A cardinal λ is a **Suslin cardinal** if there exists $A \subseteq \mathbb{R}$ such that

- A is λ-Suslin
- A is not δ-Suslin for any $\delta < \lambda$.

- ω and ω_1 are Suslin cardinals.

Lemma

Suppose $\mathbb{R}$ can be wellordered. Then the following hold.

- *c is a Suslin cardinal.*
- *c is the largest Suslin cardinal.*
- *Every (infinite) cardinal $\lambda \leq c$ is a Suslin cardinal.*

AD versus AD^+: Part I

Theorem $(\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}})$

Suppose $A \subseteq \mathbb{R}$ and A is Suslin.

- *Then $L(A, \mathbb{R}) \models \mathrm{AD}^+$.*

Theorem $(\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}})$

Let Γ be the set of all $A \subset \mathbb{R}$ such that $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then:

- *$L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$.*
- *If $\Gamma \neq \mathcal{P}(\mathbb{R})$ then $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}}$.*

- $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \neg\mathrm{AD}^+$, if consistent, is very strong.
  - It implies the consistency of $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$.

AD versus AD^+: Part II

Theorem ($\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$)

The following are equivalent.

(1) *Every set is Suslin.*

(2) $\mathrm{AD}_{\mathbb{R}}$.

(3) *Every set $A \subset \mathbb{R} \times \mathbb{R}$ can be uniformized.*

Corollary ($\mathrm{ZF} + \mathrm{DC}$)

Assume $\mathrm{AD}_{\mathbb{R}}$. Then AD^+.

Conjecture ($\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$)

Assume AD. Then AD^+.

HOD and measurable cardinals

Theorem (Solovay:1967)

Assume AD.

- *Then* ω_1^V *is a measurable cardinal in* HOD.

- Solovay's theorem gave the first connection between AD and large cardinal axioms.

Theorem

Assume AD^+ *and that* $V = L(\mathcal{P}(\mathbb{R}))$.

- *Then* ω_1^V *is the least measurable cardinal in* HOD.

- The hypothesis that $V = L(\mathcal{P}(\mathbb{R}))$ is necessary.

HOD and Woodin cardinals

Theorem $(\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}})$

One of the following hold.

(1) Θ *is a Woodin cardinal in* HOD.

(2) Θ *is a limit of Woodin cardinals in* HOD.

The Inner Model Test Question

Suppose φ is a Σ_2-sentence defining a large cardinal axiom.

- Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$. Can $\mathrm{HOD} \models \varphi$?

Lemma

Assume AD^+ *and that* $V = L(\mathcal{P}(\mathbb{R}))$.

- *Suppose* κ *is a measurable cardinal in* HOD. *Then* $\kappa < \Theta$.

Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$.

- The large cardinals of HOD occur in the interval $[\omega_1^V, \Theta]$.

Approximating Θ: The Solovay Sequence

Definition (AD^+): Solovay Sequence

$\langle \Theta_\alpha : \alpha \leq \Omega \rangle$ is the sequence defined by induction on α as follows.

1. Θ_0 is the supremum of the set of $\xi \in \mathrm{Ord}$ such that there is a surjection $\pi : \mathbb{R} \to \xi$ such that π is OD.
2. $\Theta_{\alpha+1}$ the supremum of the set of $\xi \in \mathrm{Ord}$ such that there is a surjection $\pi : \mathcal{P}(\Theta_\alpha) \to \xi$ such that π is OD.
3. $\Theta_\alpha = \sup\{\Theta_\beta \mid \beta < \alpha\}$ if α is a nonzero limit ordinal.
4. $\Theta = \Theta_\Omega$.

Theorem (Cabal)

Assume AD^+. *Then:*

- *The Suslin cardinals are closed in* Θ.
- Θ_α *is a Suslin cardinal for each* $\alpha < \Omega$.
- Θ_α *is a limit of Suslin cardinals if and only if:*
  - $\alpha > 0$ *and* α *is a limit ordinal.*

The Solovay sequence and large cardinals

Theorem (AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$)

Let $\langle \Theta_\alpha : \alpha \leq \Omega \rangle$ be the Solovay sequence.

- *Suppose that $\alpha \leq \Omega$ and that either $\alpha = 0$ or α is not a limit ordinal.*

Then the following hold.

- *Θ_α is a Woodin cardinal in HOD.*
- *Let δ be the largest Suslin cardinal such that $\delta < \Theta_\alpha$.*
  - *Then δ is a strong cardinal in $\mathrm{HOD} \cap V_{\Theta_\alpha}$.*

The Θ_α Conjecture where $\alpha > 0$ is a limit ordinal

Theorem (AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$)

Let $\langle \Theta_\alpha : \alpha \leq \Omega \rangle$ be the Solovay sequence. Suppose that $\alpha \leq \Omega$. Then the following are equivalent.

(1) *Θ_α is a Woodin cardinal in HOD.*

(2) *$\alpha = 0$ or α is a successor ordinal.*

The Θ_α Conjecture

Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$. Suppose that $\delta \leq \Theta$. Then the following are equivalent.

1. $\delta = \Theta_\alpha$ for some limit ordinal $\alpha > 0$.
2. δ is a limit of Woodin cardinals in HOD and for all $\kappa < \delta$,
    - κ is not $(<\delta)$-strong in HOD.

The Weak Θ_α Conjecture

The Weak Θ_α Conjecture

Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$. Suppose that δ is a limit of Suslin cardinals. Then the following are equivalent.

1. $\delta = \Theta_\alpha$ for some limit ordinal $\alpha > 0$.
2. δ is a limit of Woodin cardinals in HOD.

- This is a version of the Θ_α Conjecture restricted to limits of Suslin cardinals.

When there is a largest Suslin cardinal

Theorem ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$)

The following are equivalent.

(1) AD^+.

(2) *The set of Suslin cardinals is closed below* Θ.

Theorem ($\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$)

The following are equivalent.

(1) $(\neg\mathrm{AD}_{\mathbb{R}}) + \mathrm{AD}^+$.

(2) *There is a largest Suslin cardinal.*

- The models of AD^+ which satisfy $V = L(\mathcal{P}(\mathbb{R}))$, $\neg\mathrm{AD}_{\mathbb{R}}$, and in which the largest Suslin cardinal is on the Solovay Sequence are the LSA models of AD^+.

Verifying a consequence: I

Theorem (AD^+)

Assume $V = L(\mathcal{P}(\mathbb{R}))$ and that $\mathrm{AD}_{\mathbb{R}}$ does not hold.

- *Let δ be the largest Suslin cardinal.*

Assume the Weak Θ_α Conjecture. *Then the following are equivalent.*

(1) *δ is a limit of Woodin cardinals in HOD.*

(2) *$\delta = \Theta_\delta$.*

Theorem (AD^+)

Assume $V = L(\mathcal{P}(\mathbb{R}))$ and that $\mathrm{AD}_{\mathbb{R}}$ does not hold.

- *Let δ be the largest Suslin cardinal.*

Then the following are equivalent.

(1) *δ is a limit of Woodin cardinals in HOD.*

(2) *$\delta = \Theta_\delta$.*

Verifying a consequence: II

Corollary (AD^+)

Assume $V = L(\mathcal{P}(\mathbb{R}))$ and that $\mathrm{AD}_{\mathbb{R}}$ does not hold.

- *Let δ be the largest Suslin cardinal.*
- *Suppose there is a Woodin cardinal in HOD **above** δ.*

Then $\delta = \Theta_\delta$.

- The proof also adapts to yield a proof of the Weak Θ_α Conjecture.

Theorem

Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$. Suppose that δ is a limit of Suslin cardinals. Then the following are equivalent.

1. *$\delta = \Theta_\alpha$ for some limit ordinal $\alpha > 0$.*
2. *δ is a limit of Woodin cardinals in HOD.*

$V = L(\mathcal{P}(\mathbb{R}))$ versus $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$

Theorem (AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$)

For each $\gamma < \Theta$ there exists $A \subset \mathbb{R}$ such that

- $\gamma < \Theta_A$,
- $\mathrm{HOD} \cap V_{\Theta_A} = \mathrm{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A}$;

where $\Theta_A = \Theta^{L(A,\mathbb{R})}$.

- Assume AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$. Thus:
  - For the question of which Σ_2-expressible large cardinal axioms (which imply measurability) can hold in HOD, we can arguably reduce to the case that $V = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$.

Perhaps we should assume large cardinals exist in V and look for such sets $A \subseteq \mathbb{R}$ in V.

Narrowing the search: Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is **universally Baire** if:

- For all topological spaces Ω,
- For all continuous functions $\pi : \Omega \to \mathbb{R}^n$;

the preimage of A by π has the property of Baire in the space Ω.

- Universally Baire sets have the property of Baire
  - Simply take $\Omega = \mathbb{R}^n$ and π to be the identity.
- Universally Baire sets are Lebesgue measurable.

Theorem

Assume $V = L$. Then every set $A \subseteq \mathbb{R}$ is the image of a universally Baire set by a continuous function

$$F : \mathbb{R} \to \mathbb{R}.$$

The universally Baire sets and AD^+

Theorem

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.

- *Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*

Theorem

Suppose that there is a proper class of Woodin cardinals.

(1) (Martin-Steel) *Suppose $A \subseteq \mathbb{R}$ is universally Baire.*
  - *Then A is determined.*

(2) (Steel) *Suppose $A \subseteq \mathbb{R}$ is universally Baire.*
  - *Then A has a universally Baire scale.*

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire.

- *Then $L(A, \mathbb{R}) \models \mathrm{AD}^+$.*

The axiom $V = \mathrm{Ultimate}\text{-}L$

- The existence of a Woodin cardinal is expressible by a Σ_2-sentence.
  - Woodin cardinals clearly exist in V;
  - If $A \subset \mathbb{R}$ is universally Baire and there is a proper class of Woodin cardinals then

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \text{“There is a Woodin cardinal”}.$$

The axiom for $V = \mathrm{Ultimate}\text{-}L$

- There is a proper class of Woodin cardinals.
- For each Σ_2-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi.$$

Some consequences of $V = \text{Ultimate-}L$

Theorem ($V = \text{Ultimate-}L$)

The Continuum Hypothesis holds.

Theorem ($V = \text{Ultimate-}L$)

$V = \mathrm{HOD}$.

Theorem ($V = \text{Ultimate-}L$)

Let Γ^∞ be the set of all universally Baire sets $A \subseteq \mathbb{R}$. Then

$$\Gamma^\infty \neq \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R})$$

If $V = \text{Ultimate-}L$ then:

- The Axiom of Choice holds in $L(\Gamma^\infty, \mathbb{R})$.
  - This is the generalization to $V = \text{Ultimate-}L$ of the fact that if $V = L$ then there is a Σ^1_2-wellordering of the reals.

The axiom $V = \text{Ultimate-}L$ and forcing

Theorem

Suppose $M \models \mathrm{ZFC}$ is an iterable Mitchell-Steel model and there is a Woodin cardinal in M.

- *Then there exists an iterable Mitchell-Steel model N such that*
  - *$N \subseteq M$ and $N \neq M$.*
  - *$M = N[G]$ for some N-generic filter G on some $\mathbb{P} \in N$.*

- The theorem also holds for the (non-strategic) iterable models at the finite levels of supercompact.

Theorem ($V = \text{Ultimate-}L$)

V is not a nontrivial generic extension of any inner model.

Ω-logic and the Ω Conjecture: Part I

Definition (Validity in Ω-logic)

$T \models_\Omega \varphi$ if for all generic extensions $V[G]$ of V and for all α, if

$$V[G]_\alpha \models T$$

then $V[G]_\alpha \models \varphi$.

Definition

Suppose $A \subset \mathbb{R}$ is universally Baire and $M \models \mathrm{ZFC}$ is a countable transitive set. Then M is A-**closed** if

$$A \cap M[G] \in M[G]$$

for all G such that G is M-generic for some $\mathbb{P} \in M$.

Ω-logic and the Ω Conjecture: Part II

Definition (Provability in Ω-logic)

Assume there is a proper class of Woodin cardinals. Then

- $T \vdash_\Omega \varphi$

if there is a universally Baire set A such that

$$M \models \text{“}T \models_\Omega \varphi\text{”}$$

for all countable transitive $M \models \mathrm{ZFC}$ such that:

- $T \in M$,
- M is A-closed.

Definition (Ω Conjecture)

Assume there is a proper class of Woodin cardinals. Then for all sentences φ

- $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

Ω-logic and the Ω Conjecture: Part III

Theorem (Generic absoluteness of Ω-validity)

Assume there is a proper class of Woodin cardinals. Suppose T is a theory and φ is a sentence. Then the following are equivalent.

- $T \models_\Omega \varphi$.
- *For all partial orders* $\mathbb{P}$: $V^{\mathbb{P}} \models$ "$T \models_\Omega \varphi$".

Theorem (Generic absoluteness of the Ω Conjecture)

Assume there is a proper class of Woodin cardinals. Then the following are equivalent.

- Ω *Conjecture.*
- *For all partial orders* $\mathbb{P}$: $V^{\mathbb{P}} \models$ " Ω *Conjecture* ".

The Axiom $V = \mathrm{Ultimate}\text{-}L$ and the Ω Conjecture

The Θ_0 Conjecture

Assume AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$.

- Then (provably) Θ_0 is the least Woodin cardinal of HOD.

- Evidence for the Θ_0 Conjecture:

Theorem

Assume AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$. Then for a Turing cone of x:

- *Θ_0 is the least Woodin cardinal of HOD_x.*

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

Assume the Θ_0 Conjecture holds. Then the Ω Conjecture holds.

Counting Woodin cardinals in HOD

Theorem (Θ_0-Theorem)

Assume AD^+ *and* $V = L(\mathcal{P}(\mathbb{R}))$.

- *Then there is at most* 1 *Woodin cardinal in* HOD *below* Θ_0.

- Another version of the Θ_0-Theorem:

Theorem ($\Theta_{\alpha+\omega}$-Theorem)

Assume AD^+ *and* $V = L(\mathcal{P}(\mathbb{R}))$. *Let*

$$\langle \Theta_\alpha : \alpha \leq \Omega \rangle$$

be the Solovay sequence.

- *Suppose* $\alpha = 0$ *or* $\alpha = \beta + 1$ *for some* $\beta < \Omega$.

Then the following are equivalent.

1. $\Theta_{\alpha+\omega} \leq \Theta$.
2. *There are* ω*-many Woodin cardinals in* HOD *above* Θ_α.

- Θ_ω is the ω-th Woodin cardinal in HOD.

Applications of the Θ_0-Theorem

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

The Ω Conjecture holds.

- Another corollary gives a simple reformulation of the Ω Conjecture which avoids having to define $T \vdash_\Omega \varphi$, the proof relation of Ω-logic.

Notation

- *Suppose φ is a Π_2-sentence. Then*
$$V \models_\Omega \varphi$$
if φ holds in all generic extensions of V.

- $V \models_\Omega \varphi$ if and only if $\mathrm{ZC} \models_\Omega \varphi$ holds in V.

$V = \mathrm{Ultimate}\text{-}L$ versus the Ω Conjecture

Theorem

Assume there is a proper class of strong cardinals and a proper class of Woodin cardinals. Then the following are equivalent.

(1) *Ω Conjecture.*

(2) *There is a universally Baire set A with infinitely many Woodin cardinals in $\mathrm{HOD}^{L(A,\mathbb{R})}$ such that for* **all** *Π_2-sentences φ:*

- *$V \models_\Omega \varphi$ if and only if $\mathrm{HOD}^{L(A,\mathbb{R})}|\Theta^{L(A,\mathbb{R})} \models_\Omega \varphi$.*

Theorem

Assume there is a proper class of strong cardinals and a proper class of Woodin cardinals. Then the following are equivalent.

(1) *$V = \mathrm{Ultimate}\text{-}L$.*

(2) *There is a universally Baire set A with infinitely many Woodin cardinals in $\mathrm{HOD}^{L(A,\mathbb{R})}$ such that for* **all** *Π_2-sentences φ:*

- *$V \models \varphi$ if and only if $\mathrm{HOD}^{L(A,\mathbb{R})}|\Theta^{L(A,\mathbb{R})} \models \varphi$.*