Two applications of recursion theory to set theory

Liang Yu

Mathematical Department
Nanjing University

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Spector-Gandy's theorem

Theorem (Spector, Gandy)

A set A of reals is Π^1_1 if and only if there is a Σ_1 formula φ so that $\forall x(x \in A \leftrightarrow L_{\omega_1^x}[x] \models \varphi(x))$.

Δ^1_1-randomness

Definition

A real r is Δ^1_1-random if it does not belong to any Δ^1_1-null set.

Proposition (Sacks)

- *If r is $\Delta^1_1(\mathcal{O})$-random, then $\omega_1^r = \omega_1^{\mathrm{CK}}$.*
- *Any Σ^1_1-null set is covered by a Δ^1_1-null set.*

Kurtz randomness

Definition

A real is *Kurtz random* if it does not belong to any Π^0_1-null set.

Theorem (Johanna and Stephan)

If x is Kurtz random and y is x-Kurtz random, then $x \oplus y$ is also Kurtz random.

Theorem (Kjos-Hanssen, Nies, Stephan and Y)

For any real x and $y \geq_T x'$, there is an x-Kurtz random $r \equiv_T y$.

Martin's theorem

Theorem (Martin)

If A is an uncountable Δ^1_1, then A ranges over the upper cone $\geq_h \mathcal{O}$.

Luzin-(N)-Property

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Proposition (Pauly, Westrick, Y)

If $f : \mathbb{R} \to \mathbb{R}$ is a recursive function, then for any x and $\Delta^1_1(x)$-random real r, $f^{-1}(r)$ only contains $\Delta^1_1(x)$-random real.

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So from a recursion theoretical point view, a function having Luzin-(N)-Property means that it does not increase randomness.

Banach's theorem

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Theorem (Russian school; Louveau)

If f is has property (N) and measurable, then it has property T_2.

A recursion theoretical proof of Banach's theorem

Let f be a recursive function having property N. Let r be a $\Delta^1_1(\mathcal{O})$-random real. Then $f^{-1}(r)$ only contains $\Delta^1_1(\mathcal{O})$-random. So $f(x) = r$ implies $\omega_1^x = \omega_1^{\mathrm{CK}}$. Note that $f^{-1}(r)$ is a $\Delta^1_1(r)$ set and only contains reals Turing computing r. By Martin's theorem, $f^{-1}(r)$ is countable.

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To prove the second theorem, note that a measurable function f equals to a Borel function almost everywhere. Since f has property-(N), we may ignore the null part. Then apply the method above to the Borel function.

Some more results (1)

Definition

A function f has *Luzin-(M)-Property* if it maps a null set to a countable set.

Theorem (Pauly, Westrick, Y)

- *If f is a continuous function and has property (M), then f is a constant function.*
- *If f is has property (M) and measurable, then the range of f is countable.*

Some more results (2)

Proof.

We only prove (1) for a recursive function f. For any x, let g be $\Delta^1_1(\mathcal{O}^x)$-generic. Then x and g forma a minimal pair of Δ^1_1-degrees. Moreover, x belongs to a $\Delta^1_1(g)$-null set A. So $f(A)$ is a $\Sigma^1_1(g)$ and countable set. Thus $f(x) \leq_{\Delta^1_1} g, x$ and so must be Δ^1_1. In other words, the range of f is countable and so constant. □

Some questions

Question

(1) *Is there a function having property-(N) but not having property T_2?*

(2) *Is there an uncountable ideal I of Turing degrees so that for any real x and almost every real r, r is random relative to the ideal generated by $I \cup \{x\}$?*

Both questions have positive answers under certain set theory assumptions.

On $\sigma(J)$

Given a collection $\mathcal{A}$ of sets of reals, we use $\sigma(A)$ to denote the σ-algebra generated by $\mathcal{A}$.

Jordan measurable sets

Definition

A set of reals is *Jordan measurable* if its characteristic function is Riemann integrable.

We use J to denote the collection of Jordan measurable sets.

On $\sigma(J)$

Given a collection $\mathcal{A}$ of sets of reals, we use $\sigma(A)$ to denote the σ-algebra generated by $\mathcal{A}$.

Johnson's theorem

Theorem (Johnson)

A set A belongs to $\sigma(J)$ if and only if there is a Borel set $B \subseteq A$ and an F_σ null set X so that $A \subseteq B \cup X$.

Johnson's question

Question (Johnson)

Where does $\sigma(J)$ stand relative to analytic-sets in $[0,1]$?

The question was also asked by Arnold Miller.

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Maudin answered this question by showing that there is an analytic set which does not belong to $\sigma(J)$.

Constructibility and the largest Π^1_1-thin set

A set is *thin* if it has no perfect subset.

Theorem (Forklore)

$\mathcal{C} = \{x \mid x \in L_{\omega_1^x}\}$ *is the largest* Π^1_1*-thin set.*

A recursion theoretical solution (1)

Theorem

Suppose that every real is constructible, then $\mathcal{C}$ does not belong to $\sigma(J)$.

Proof.

Suppose that $\mathcal{C} = B \cup X$ for some Borel set B and a set X which is a subset of an F_σ null-set. Then

1. B must be countable.
2. There is a real x so that X contains no x-Kurtz random real.
3. Now, by KNSY's result, fix a real $r \in \mathcal{C}$ which is x-Kurtz random.
4. Then $r \notin B \cup X$, a contradiction.

□

A recursion theoretical solution (2)

Theorem

$\mathcal{D} = \{x \oplus y \mid y \in L_{\omega_1^{x\oplus y}}[x]\}$ *does not belong to* $\sigma(J)$.

Proof.

Suppose that $\mathcal{D} = B \cup X$ for some $\Delta^1_1(x)$ set B and a set X which is a subset of a $\Sigma^0_2(x)$ null-set for some x. By KNSY's result, we may pick up some $x_0 \geq_T x'$ which is x-Kurtz random.

1. $B \cap \{x_0 \oplus y \mid x_0 \oplus y \in \mathcal{D}\}$ must be countable in $L[x_0]$.
2. By KNSY's result again, pick up some x_0-Kurtz random real $y \geq_T x_0'$ so that $x_0 \oplus y \in \mathcal{D} \setminus B$.
3. Now, JS's result, $x_0 \oplus y$ is x-Kurtz random and so does not belong to X.
4. Then $x_0 \oplus y \notin B \cup X$, a contradiction.

Mauldin's question

A σ-ideal is an idea which is closed under countably many union.

Question

For what σ-ideals one can derive similar results?

Borel generated σ-ideals

A set A can be approximated by a σ-idea I if there is a Borel set B and an element $X \in I$ so that $A = B \cup X$.

Definition

A σ-idea I is *Borel generated* if for any $X \in I$, there is some Borel set $Y \in I$ so that $X \subseteq Y$.

Theorem

Suppose that every real is constructible. Then for any Borel generated σ-ideal I, the followings are equivalent:

- (1) *Every Σ^1_1-set can be approximated by I;*
  (2) *There is some $X \in I$ containing an upper cone of hyperdegrees.*
- (1) *Every analytic set can be approximated by I;*
  (2) *For any x, there is some $X \in I$ and $x_0 \geq_T x$ such that $\{z \mid z \oplus x \geq_h x_0\} \subseteq X$.*

Thanks