An EM blueprint for $L[T_3]$

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Assume AD throughout this talk.

$0^{\#}$ is the unique wellfounded remarkable EM blueprint.

$\mathcal{L}_0 = \{\underline{\in}\}$. $\mathcal{L} = \mathcal{L}_0 \cup \{\underline{c_n} : n \in \omega\}$.

Definition 0.1. An EM blueprint is a complete consistent $\mathcal{L}$-theory containing the following axioms:

1. ZFC.
2. $V = L$.
3. $\varphi(\underline{c_1}, \dots, \underline{c_n}) \to \varphi(\underline{c_{f(1)}}, \dots, \underline{c_{f(n)}})$, where $\varphi(x_1, \dots, x_n)$ is an $\mathcal{L}_0$-formula, f is order preserving.
4. $c_0 \in \text{Ord}$.

5. $\underline{c_0} < \underline{c_1}$.

Definition 0.2. An EM blueprint Γ is *unbounded* iff it contains

$$\tau(\underline{c_1}, \ldots, \underline{c_n}) \in \mathrm{Ord} \rightarrow \tau(\underline{c_1}, \ldots, \underline{c_n}) < \underline{c_{n+1}}.$$

for any $\mathcal{L}_0$-Skolem term τ.

Definition 0.3. An EM blueprint Γ is *remarkable* iff it contains

$$\tau(\underline{c_1}, \ldots, \underline{c_n}, \underline{c_{n+1}}, \ldots, \underline{c_{n+m}}) < \underline{c_{n+1}}$$
$$\rightarrow \tau(\underline{c_1}, \ldots, \underline{c_n}, \underline{c_{n+1}}, \ldots, \underline{c_{n+m}}) = \tau(\underline{c_1}, \ldots, \underline{c_n}, \underline{c_{n+m+1}}, \ldots, \underline{c_{n+2m}}).$$

for any $\mathcal{L}_0$-Skolem term τ.

Definition 0.4. If $\alpha \in \mathrm{Ord}$, $\mathcal{M}_{\Gamma,\alpha}$ is the unique (up to isomorphism) model of ZFC+V=L such that $\mathcal{M}_{\Gamma,\alpha}$ is the Skolem hull of $\{c_i : i \in \alpha\}$ and for any $c_0 < c_1 < \ldots$, $(\mathcal{M}_{\Gamma,\alpha}, \in, c_0, c_1, \ldots) \models \Gamma$.

Definition 0.5. An EM blueprint Γ is *wellfounded* iff for any $\alpha \in \mathrm{Ord}$, $\mathcal{M}_{\Gamma,\alpha}$ is wellfounded.

Wellfoundedness is a Π^1_2 property.

Definition 0.6. $0^\#$ is the unique remarkable, wellfounded EM blueprint.

Definition 0.7. A *level-1 tree* is a set $P \subseteq \omega^{<\omega}$ such that

1. $0 \notin A$.
2. If $(i_1, \ldots, i_{k+1}) \in P$, $k \geq 1$, then $(i_1, \ldots, i_k) \in P$ and for any $j < i_{k+1}$, $(i_1, \ldots, i_k, j \in P)$.

Nodes in a level-1 tree are ordered by $<_{BK}$. If $s \subsetneq t$, then $t <_{BK} s$. A level-1 tree P is *regular* iff $(1) \in P$. So if P is regular and $P \neq \emptyset$, then $(0) = \max P$. The *ordinal representation* of P is

$$\operatorname{rep}(P) = \{(p) : p \in P\} \cup \{(p, n) : p \in P, n < \omega\}.$$

$\operatorname{rep}(P)$ is endowed with the ordering

$$<^P = <_{BK} \restriction \operatorname{rep}(P).$$

Thus, (p) is the $<^P$-supremum of (p, n) for $n < \omega$.

If $B \subseteq \omega_1$, $B^{P\uparrow}$ is the set of $f : \operatorname{rep}(P) \to B$ which are continuous, order preserving. If $f \in \omega_1^{P\uparrow}$, let

$$[f]^P = ([f]^P_p)_{p \in P}$$

where $[f]^P_p = f((p))$ for $p \in P$. Let $[B]^{P\uparrow} = \{[f]^P : f \in B^{P\uparrow}\}$. P is Π^1_1-*wellfounded* iff $P \cup \{\emptyset\}$ is a wellfounded tree. A tuple $\vec{\alpha} = (\alpha_p)_{p \in P}$ *respects* P iff $\vec{\alpha} \in [\omega_1]^{P\uparrow}$ iff the map $p \mapsto \alpha_p$ is an isomorphism between $(P, <_{BK})$ and $(\{\alpha_p : p \in P\}, <)$.

Definition 0.8. A *finite level-1 tower* is a tuple $(P_i)_{i \leq n}$ such that $n < \omega$, P_i is a level-1 tree of card i for any i, and $i < j \to P_i \subseteq P_j$.

Definition 0.9. Suppose P is a level-1 tree. The set of P-*descriptions* is

$$\mathrm{desc}(P) = P \cup \{\emptyset\}.$$

The *constant P-description* is $\emptyset$. $p \prec_P p'$ iff $p, p' \in \mathrm{desc}(P)$ and $p <_{BK} p'$.

Definition 0.10. Suppose P, W are level-1 trees. A function $\sigma : P \cup \{\emptyset\} \to W \cup \{\emptyset\}$ *factors* (P, W) iff $\sigma(\emptyset) = \emptyset$ and σ preserves the $<_{BK}$-order (not necessarily the tree order). σ *factors* $(P, *)$ iff σ factors (P, W) for some level-1 tree W.

Fact 0.11. Suppose P, W are Π^1_1-wellfounded. Then $\mathrm{o.t.}(<^P) \leq \mathrm{o.t.}(<^W)$ iff $\exists \sigma(\sigma \text{ factors } (P, W))$.

If P is a finite level-1 tree, μ^P is a measure on $[\omega_1]^{P\uparrow}$. $A \in \mu^P$ iff there is club C such that $[C]^{P\uparrow} \subseteq A$. j^P is the ultrapower map of μ^P. If $p \in \mathrm{desc}(P)$, then

$$\mathrm{seed}^P_p = \begin{cases} [\vec{\alpha} \to \alpha_p]_{\mu^P}, & \text{if } p \in P \\ j^P(\omega_1), & \text{if } p = \emptyset. \end{cases}$$

Put $\mathrm{seed}^P = (\mathrm{seed}^P_p)_{p \in \mathrm{desc}(P)}$. So $p \prec^P p'$ iff $\mathrm{seed}^P_p < \mathrm{seed}^P_{p'}$.

If $P \subseteq P'$, both finite level-1 trees, then $\mu^{P'}$ projects to μ^P, i.e., the identity map factors (P, P'). Let $j^{P,P'}$ be the factoring map.

A *partial level* ≤ 1 *tree* is a pair (P, t) such that P is a finite regular level-1 tree and either

1. $t \notin P \wedge P \cup \{t\}$ is a regular level-1 tree, or
2. $P \neq \emptyset$, $t = -1$ (-1 is the "level-0 component").

(P, t) is of degree 0 if $t = -1$, of degree 1 otherwise. $\mathrm{dom}(P, t) = P \cup \{t\}$. $\vec{\alpha}$ respects (P, t) iff $\vec{\alpha} \restriction P$ respects P, $t = -1 \to \alpha_t < \omega$, $t \neq -1 \to \vec{\alpha}$ respects $P \cup \{t\}$. The cardinality of (P, t) is $\mathrm{card}(P, t) = \mathrm{card}(P) + 1$. The unique partial level ≤ 1 tree of card 1 is $(\emptyset, (0))$. If (P, t) of deg 1, its *completion* is $P \cup \{t\}$. $(P, -1)$ has no completion. (P, t) is a partial subtree of P' iff the completion of (P, t) exists and is a subtree of P'.

A *partial level ≤ 1 tower of discontinuous type* is a nonempty finite sequence $(\vec{P}, \vec{p}) = (P_i, p_i)_{i \leq k}$ such that $\mathrm{card}(P_0, p_0) = 1$, each (P_i, p_i) is a partial level ≤ 1 tree, and P_{i+1} is the completion of (P_i, p_i). A *partial level ≤ 1 tower of continuous type* is $(P_i, p_i)_{i<k}{}^\frown(P_*)$ such that either $k = 0 \wedge P_* = \emptyset$ or $(P_i, p_i)_{i<k}$ is a partial level ≤ 1 tower of discontinuous type $\wedge P_*$ is the completion of (P_{k-1}, p_{k-1}).

For notational convenience, the information of a partial level ≤ 1 tower is compressed into a potential partial level ≤ 1 tower. We say a *potential partial level ≤ 1 tower* is $(P_*, \vec{p}) = (P_*, (p_i)_{i<\mathrm{lh}(\vec{p})})$ such that for some level-1 tower $\vec{P} = (P_i)_{i \leq k}$, either $P_* = P_k \wedge (\vec{P}, \vec{p})$ is a partial level ≤ 1 tower of discontinuous type or $(\vec{P}, \vec{p})^\frown(P_*)$ is a partial level ≤ 1 tower of continuous type. If $(P_*, (p_i)_{i \leq k})$ is a potential partial level ≤ 1 tower of discontinuous type, its *completion* is the completion of (P_*, p_k).

A *tree of level-1 trees* is a tree T on $\omega^{<\omega}$ (i.e., $T \subseteq (\omega^{<\omega})^{<\omega}$ and closed under $\subseteq$) and such that for any $s \in T$, $\{a \in \omega^{<\omega} : s^\frown(a) \in T\}$ is a level-1 tree.

Definition 0.12. A *level-2 tree* is a function Q such that $\mathrm{dom}(Q)$ is a tree of level-1 trees, $\emptyset \in \mathrm{dom}(Q)$ and for any $q \in \mathrm{dom}(Q)$, $(Q(q \upharpoonright l))_{l \leq \mathrm{lh}(q)}$ is a partial level ≤ 1 tower of discontinuous type. In particular, $Q(\emptyset) = (\emptyset, (0))$.

We denote $Q(q) = (Q_{\mathrm{tree}}(q), Q_{\mathrm{node}}(q))$ and $Q[q] = (Q_{\mathrm{tree}}(q), (Q_{\mathrm{node}}(q \upharpoonright l))_{l \leq \mathrm{lh}(q)})$. So $Q[q]$ is a potential partial level ≤ 1 tower of discontinuous type. Denote $Q\{q\} = \{a \in \omega^{<\omega} : q^\frown(a) \in \mathrm{dom}(Q)\}$, which is a level-1 tree. The *cardinality* of Q is $\mathrm{card}(Q) = \mathrm{card}(\mathrm{dom}(Q))$. $\mathrm{card}(Q)$ could be finite or $\aleph_0$.

Definition 0.13. Q^0, Q^1, Q^{20}, Q^{21} denote the following typical level ≤ 2 trees of cardinalities at most 2:

- ${}^1Q^0 = \emptyset$, ${}^1Q^1 = \{(0)\}$, $\mathrm{dom}({}^2Q^0) = \mathrm{dom}({}^2Q^1) = \{\emptyset\}$.
- For $d \in \{0, 1\}$, ${}^1Q^{2d} = \emptyset$, $\mathrm{dom}({}^2Q^{2d}) = \{\emptyset, ((0))\}$, ${}^2Q^{2d}((0))$ is of degree d.

For Q a level-2 tree, Let

$$\mathrm{dom}^*(Q) = \mathrm{dom}(Q) \cup \{q^\frown(-1) : q \in \mathrm{dom}(Q)\}.$$

Here -1 is a distinguished element which is $<_{BK}$-smaller than any node in $\omega^{<\omega}$. So $<_{BK}\restriction \mathrm{dom}^*(Q)$ extends $<_{BK}\restriction \mathrm{dom}(Q)$ where $q^\frown(-1)$ comes before any $q^\frown(s) \in \mathrm{dom}(Q)$. If $q \neq \emptyset$, denote $Q\{q,-\} = \{q^{-\frown}(-1)\} \cup \{q^{-\frown}(a) : Q_{\mathrm{tree}}(q^{-\frown}(a)) = Q_{\mathrm{tree}}(q) \wedge a <_{BK} q(\mathrm{lh}(q)-1)\}$, $Q\{q,+\} = \{q^-$ $\{q^{-\frown}(a) : Q_{\mathrm{tree}}(q^{-\frown}(a)) = Q_{\mathrm{tree}}(q) \wedge a >_{BK} q(\mathrm{lh}(q)-1)\}$. For $q \in \mathrm{dom}^*($ q is *of discontinuous type* if $q \in \mathrm{dom}(Q)$; q is *of continuous type* if $q \in \mathrm{dom}^*(Q) \setminus \mathrm{dom}(Q)$. In particular, $\{\emptyset, (-1)\} \subseteq \mathrm{dom}^*(Q)$. Put $Q[q^\frown(-1)] = (P, (Q_{\mathrm{node}}(q \restriction l))_{l \leq \mathrm{lh}(q)})$, where P is the completion of $Q(q)$. So $Q[q^\frown(-1)]$ is a potential partial level ≤ 1 tower of continuous type.

Definition 0.14. Suppose Q is a level-2 tree. A *Q-description* is a triple

$$\mathbf{q} = (q, P, \vec{p})$$

such that $q \in \mathrm{dom}^*(Q)$ and $(P, \vec{p}) = Q[q]$. $\mathrm{desc}(Q)$ is the set of Q-descriptions. A Q-description $(q, P, \vec{p})$ is of *(dis-)continuous type* iff q is of (dis-)continuous type. The *constant Q-description* is $(\emptyset, \emptyset, (0))$.

Definition 0.15. Suppose Q is a level ≤ 2 tree. An *extended Q-description* is either a Q-description or of the form $(2,(q,P,\vec{p}))$ such that $(2,(q^\frown(-1),P,\vec{p}))$ is a Q-description of continuous type. $\operatorname{desc}^*(Q)$ is the set of extended Q-descriptions. $(d,\mathbf{q}) \in \operatorname{desc}^*(Q)$ is *regular* iff either $(d,\mathbf{q}) \in \operatorname{desc}(Q)$ of discontinuous type or $(d,\mathbf{q}) \notin \operatorname{desc}(Q)$.

If $\mathbf{q} = (q,P,\vec{p}) \in \operatorname{desc}(Q)$ is of discontinuous type, put $\mathbf{q}^\frown(-1) = (q^\frown(-1), P$ where P^+ is the completion of $(P,\vec{p})$. If $\vec{\alpha} = (\alpha_p)_{p\in N}$ is a tuple indexed by N, $q \in \operatorname{dom}^*(Q)$, $\operatorname{dom}(Q(q^-)) \subseteq N$ if $q \neq \emptyset$, we put

$$\vec{\alpha} \oplus_Q q = (\alpha_{p_0}, q(0), \dots, \alpha_{p_{\operatorname{lh}(q)-1}}, q(\operatorname{lh}(q)-1)),$$

where $p_i = Q_{\text{node}}(q \restriction i)$. The ordinal representation of Q is the set

$$\begin{aligned}\operatorname{rep}(Q) = & \{\vec{\alpha} \oplus_Q q : q \in \operatorname{dom}(Q), \vec{\alpha} \text{ respects } Q_{\text{tree}}(q)\} \\ & \cup \{\vec{\alpha} \oplus_Q q^\frown(-1) : q \in \operatorname{dom}(Q), \vec{\alpha} \text{ respects } Q(q)\}.\end{aligned}$$

$\operatorname{rep}(Q)$ is endowed with the $<_{BK}$ ordering:

$$<^Q = <_{BK} \restriction \operatorname{rep}(Q).$$

When Q is finite, $<^Q$ has ot $\omega_1 + 1$ with maximum $\emptyset$.

If $B \subseteq \omega_1$, we put

$$f \in B^{Q\uparrow}$$

iff f is an order preserving, continuous function from $\mathrm{rep}(Q)$ to $B \cup \{\omega_1\}$. If $f \in B^{Q\uparrow}$, for each $q \in \mathrm{dom}(Q)$, letting $P_q = Q_{\mathrm{tree}}(q)$, f_q is the function on $[\omega_1]^{P_q\uparrow}$ that sends $\vec{\alpha}$ to $f(\vec{\alpha} \oplus_Q q)$, and

$$[f]^Q = ([f]^Q_q)_{q \in \mathrm{dom}(Q)}$$

where $[f]^Q_q = [f_q]_{\mu^{P_q}}$.

If $y \in [\mathrm{dom}(Q)]$, let $Q(y) =_{\mathrm{DEF}} \cup_{n<\omega} Q_{\mathrm{tree}}(y \restriction n)$ be an infinite level-1 tree. Q is Π^1_2-*wellfounded* iff

1. $\forall q \in \mathrm{dom}(Q)$ $Q\{q\}$ is Π^1_1-wellfounded,
2. $\forall y \in [\mathrm{dom}(Q)]$ $Q(y)$ is not Π^1_1-wellfounded.

A level-2 tree Q is a called a *subtree* of Q' iff Q is a subfunction of Q'. A *finite level-2 tower* is a (possibly empty) sequence $(Q_i)_{1 \leq i \leq n}$ such that Q_i is a level-2 tree for $1 \leq i \leq n$, $\mathrm{card}(Q_i) = i$ and $i < j \to Q_i$ is a subtree of Q_j.

A *level* ≤ 2 *tree* is a pair $Q = ({}^1 Q, {}^2 Q)$ such that ${}^d Q$ is a level-d tree for $d \in \{1,2\}$. Its *cardinality* is $\mathrm{card}(Q) = \sum_d \mathrm{card}({}^d Q)$. We follow the convention that ${}^d Q$ always stands for the level-d component of a level ≤ 2 tree Q. Q is a *level* ≤ 2 *subtree* of Q' iff ${}^d Q$ is a level-d subtree of ${}^d Q'$ for $d \in \{1,2\}$. $\mathrm{rep}(Q) = \bigcup_d (\{d\} \times \mathrm{rep}({}^d Q))$. $<^Q = <_{BK} \restriction \mathrm{rep}(Q)$. So $<^Q$ is essentially the concatenation of $<^{{}^1 Q}$ and $<^{{}^2 Q}$. $\mathrm{dom}(Q) = \bigcup_d (\{d\} \times \mathrm{dom}({}^d Q))$, $\mathrm{dom}^*(Q) = \bigcup_d (\{d\} \times \mathrm{dom}^*({}^d Q))$, where $\mathrm{dom}^*({}^1 Q) = \mathrm{dom}({}^1 Q) = {}^1 Q$. $\mathrm{desc}(Q) = \bigcup_d (\{d\} \times \mathrm{desc}({}^d Q))$ is the set of Q-descriptions. $(d, \mathbf{q}) \in \mathrm{desc}(Q)$ is *of continuous type* iff $d = 2$ and $\mathbf{q}$ is of continuous type; otherwise, $(d, \mathbf{q})$ is *of discontinuous type*. Q is Π^1_2-*wellfounded* iff ${}^1 Q$ is Π^1_1-wellfounded and ${}^2 Q$ is Π^1_2-wellfounded.

If f is a function on $\mathrm{rep}(Q)$, let ${}^d f$ be the function on $\mathrm{rep}({}^d Q)$ that sends v to $f(d, v)$. If B is a subset of ω_1, we put

$$f \in B^{Q\uparrow}$$

iff f is an order preserving, continuous function on $\mathrm{rep}(Q)$, and ${}^d f \in B^{{}^d Q \uparrow}$

for $d \in \{1,2\}$. f represents a card(Q)-tuple of ordinals

$$[f]^Q = ({}^d[f]^Q_q)_{(d,q)\in \mathrm{dom}(Q)}$$

where ${}^d[f]^Q_q = [{}^d f]^{{}^dQ}_q$. In particular, we must have ${}^2[f]^Q_\emptyset = \omega_1$. Let

$$[B]^{Q\uparrow} = \{[f]^Q : f \in B^{Q\uparrow}\}.$$

Suppose $(2,\mathbf{q}) = (2,(q,P,\vec{p})) \in \mathrm{desc}^*(Q)$. If $f \in \omega_1^{Q\uparrow}$, ${}^2 f_{\mathbf{q}}$ is the function on $[\omega_1]^{P\uparrow}$ defined as follows: ${}^2 f_{\mathbf{q}} = {}^2 f_q$ if $(2,\mathbf{q}) \in \mathrm{desc}(Q)$; ${}^2 f_{\mathbf{q}}(\vec{\alpha}) = {}^2 f_q(\vec{\alpha} \restriction {}^2Q_{\mathrm{tree}}(q))$ if $(2,\mathbf{q}) \notin \mathrm{desc}(Q)$. If $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in\mathrm{dom}(Q)} \in [\omega_1]^{Q\uparrow}$, we define ${}^d\beta_{\mathbf{q}}$ for $(d,\mathbf{q}) \in \mathrm{desc}^*(Q)$: if $d = 2$, $\mathbf{q} = (q,P,\vec{p})$, put ${}^d\beta_{\mathbf{q}} = [{}^d f_{\mathbf{q}}]_{\mu^P}$ where $\vec{\beta} = [f]^Q$. Clearly, ${}^2\beta_{\mathbf{q}} = {}^2\beta_q$ if $(2,\mathbf{q}) \in \mathrm{desc}(Q)$ of discontinuous type, ${}^2\beta_{\mathbf{q}} = j^{{}^2Q_{\mathrm{tree}}(q),P}({}^2\beta_q)$ if $(2,\mathbf{q}) \notin \mathrm{desc}(Q)$.

Let Q be a finite level ≤ 2 tree. We define

$$A \in \mu^Q$$

iff there is a club C such that

$$[C]^{Q\uparrow} \subseteq A.$$

j^Q is the ultrapower map of μ^Q. If Q is a subtree of Q', both finite, then $\mu^{Q'}$ projects μ^Q and $j^{Q,Q'}$ is the factoring map.

Definition 0.16. A *partial level ≤ 2 tree* is a pair $(Q,(d,q,P))$ such that Q is a finite level ≤ 2 tree, and one of the following holds:

1. $(d,q,P) = (0,-1,\emptyset)$, or
2. $d = 1$, $q \notin {}^1Q$, ${}^1Q \cup \{q\}$ is a level-1 tree, $P = \emptyset$, or
3. $d = 2$, $q \notin \mathrm{dom}({}^2Q)$, $\mathrm{dom}({}^2Q) \cup \{q\}$ is tree of level-1 trees, P is the completion of ${}^2Q(q^-)$. (In particular, ${}^2Q(q^-)$ must have degree 1.)

The *degree* of $(Q,(d,q,P))$ is d. We put $\mathrm{dom}(Q,(d,q,P)) = \mathrm{dom}(Q) \cup \{(d,q)\}$. The *cardinality* of $(Q,(d,q,P))$ is $\mathrm{card}(Q,(d,q,P)) = \mathrm{card}(Q)+1$.

The *uniform cofinality* of a partial level ≤ 2 tree $(Q,(d,q,P))$ is

$$\mathrm{ucf}(Q,(d,q,P)),$$

defined as follows.

1. $\mathrm{ucf}(Q,(d,q,P)) = (0,-1)$ if $d = 0$;

2. $\operatorname{ucf}(Q,(d,q,P)) = (1, q^-)$ if $d = 1$, $\operatorname{lh}(q) > 1$;

3. $\operatorname{ucf}(Q,(d,q,P)) = (2, (\emptyset, \emptyset, (0)))$ if $d = 1$, $\operatorname{lh}(q) = 1$;

4. $\operatorname{ucf}(Q,(d,q,P)) = (2, (q', P, \vec{p}))$ if $d = 2$, ${}^2Q[q'] = (P, \vec{p})$, and q' is the $<_{BK}$-least element of ${}^2Q\{q,+\}$, $q' \neq q^-$;

5. $\operatorname{ucf}(Q,(d,q,P)) = (2, (q^-, P, \vec{p}))$ if $d = 2$, ${}^2Q[q^-] = (P^-, \vec{p})$, and ${}^2Q\{q,+\} = \{q^-\}$.

So $\operatorname{ucf}(Q,(d,q,P))$ is either $(0,-1)$ or a regular extended Q-description. The *cofinality* of $(Q,(d,q,P))$ is

$$\operatorname{cf}(Q,(d,q,P)) = \begin{cases} 0 & \text{if } d = 0, \\ 1 & \text{if } d = 1 \text{ and } q = \min(\prec^{1Q \cup \{q\}}), \\ 2 & \text{otherwise.} \end{cases}$$

A tuple $\vec{\beta} = ({}^e\beta_t)_{(e,t) \in \operatorname{dom}(Q,(d,q,P))}$ *respects* $(Q,(d,q,P))$ iff $\vec{\beta} \restriction \operatorname{dom}(Q)$ respects Q and ${}^d\beta_q < \omega$ if $d = 0$, $\vec{\beta}$ respects a completion of $(Q,(d,q,P))$ otherwise. A partial level ≤ 2 tree of degree 0 has no completion. A *completion* of a partial level ≤ 2 tree $(Q,(d,q,P))$ of degree ≥ 1 is a level ≤ 2

tree Q^* such that $\operatorname{dom}(Q^*) = \operatorname{dom}(Q, (d, q, P))$, ${}^2Q^* \restriction \operatorname{dom}({}^2Q) = {}^2Q$, and either $d = 1$ or $d = 2 \wedge {}^2Q_{\text{tree}}(t) = P$. For a level ≤ 2 tree Q', $(Q, (d, q, P))$ is a *partial subtree* of Q' iff a completion of $(Q, (d, q, P))$ is a subtree of Q'.

A *partial level ≤ 2 tower of discontinuous type* is a nonempty finite sequence $(Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ such that $\operatorname{card}(Q_1) = 1$, each $(Q_i, (d_i, q_i, P_i))$ is a partial level ≤ 2 tree, and each Q_{i+1} is a completion of $(Q_i, (d_i, q_i, P_i))$. A *partial level ≤ 2 tower of continuous type* is $(Q_i, (d_i, q_i, P_i))_{1 \leq i < k}{}^\frown(Q_*)$ such that either $k = 0 \wedge Q_*$ is the level ≤ 2 tree of cardinality 1 or $(Q_i, (d_i, q_i, P_i))$ is a partial level ≤ 2 tower of discontinuous type $\wedge Q_*$ is a completion of $(Q_{k-1}, (d_{k-1}, q_{k-1}, P_{k-1}))$. For notational convenience, the information of a partial level ≤ 2 tower is compressed into a potential partial level ≤ 2 tower.

A *potential partial level ≤ 2 tower* is $(Q_*, \overrightarrow{(d, q, P)}) = (Q_*, (d_i, q_i, P_i)_{1 \leq i \leq \text{lh}}$ such that for some level ≤ 2 tower $\vec{Q} = (Q_i)_{1 \leq i \leq k}$, either $Q_* = Q_k \wedge (\vec{Q}, \overrightarrow{(d, q, P)})$ is a partial level ≤ 2 tower of discontinuous type or $(\vec{Q}, \overrightarrow{(d, q, P)})^\frown$ is a partial level ≤ 2 tower of continuous type.

Definition 0.17. A *level-3 tree of uniform cofinality*, or *level-3 tree*, is a function

$$R$$

such that $\emptyset \notin \mathrm{dom}(R)$, $\mathrm{dom}(R) \cup \{\emptyset\}$ is tree of level-1 trees and for any $r \in \mathrm{dom}(R)$, $(R(r \restriction l))_{1 \leq l \leq \mathrm{lh}(r)}$ is a partial level ≤ 2 tower of discontinuous type. If $R(r) = (Q_r, (d_r, q_r, P_r))$, we denote $R_{\mathrm{tree}}(r) = Q_r$, $R_{\mathrm{node}}(r) = (d_r, q_r)$, $R[r] = (Q_r, (d_{r \restriction l}, q_{r \restriction l}, P_{r \restriction l})_{1 \leq l \leq \mathrm{lh}(r)})$. $R[r]$ is a potential partial level ≤ 2 tower of discontinuous type. If Q is a completion of $R(r)$, put $R[r, Q] = (Q, (d_{r \restriction l}, q_{r \restriction l}, P_{r \restriction l})_{1 \leq l \leq \mathrm{lh}(r)})$, which is a potential partial level ≤ 2 tower of continuous type. For $r \in \mathrm{dom}(R) \cup \{\emptyset\}$, put $R\{r\} = \{a \in \omega^{<\omega} : r^\frown(a) \in \mathrm{dom}(R)$ which is a level-1 tree.

The *cardinality* of R is $\mathrm{card}(R) = \mathrm{card}(\mathrm{dom}(R))$. R is said to be *regular* iff $((1)) \notin \mathrm{dom}(R)$. In other words, when $R \neq \emptyset$, $((0))$ is the $<_{BK}$-maximum of $\mathrm{dom}(R)$.

Suppose R is a level-3 tree. Let $\mathrm{dom}^*(R) = \mathrm{dom}(R) \cup \{r^\frown(-1) : r \in \mathrm{dom}(R)\}$ For $r \in \mathrm{dom}(R)$, put $R\{r, -\} = \{r^{-\frown}(-1)\} \cup \{r^{-\frown}(a) : R_{\mathrm{tree}}(r^{-\frown}(a)) = R_{\mathrm{t}}$
$R\{r, -\} = \{r^-\} \cup \{r^{-\frown}(a) : R_{\mathrm{tree}}(r^{-\frown}(a)) = R_{\mathrm{tree}}(r), a >_{BK} r(\mathrm{lh}(r) - 1)\}$

If $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in N}$ is a tuple indexed by N, $r \in \mathrm{dom}^*(R)$, $\mathrm{lh}(r) = k$, either $k = 1$ or $\mathrm{dom}(R(r^-)) \subseteq N$, we put

$$\vec{\beta} \oplus_R r = (r(0), {}^{d}\beta_{q_1}, r(1), \ldots, {}^{d}\beta_{q_{k-1}}, r(k-1)),$$

where $(d_i, q_i) = R_{\mathrm{node}}(r \restriction i)$.

The *ordinal representation* of R is the set

$$\begin{aligned}\mathrm{rep}(R) =& \{\vec{\beta} \oplus_R r : r \in \mathrm{dom}(R), \vec{\beta} \text{ respects } R_{\mathrm{tree}}(r)\} \\ & \cup \{\vec{\beta} \oplus_R r ^\frown (-1) : r \in \mathrm{dom}(R), \vec{\beta} \text{ respects } R(r)\}.\end{aligned}$$

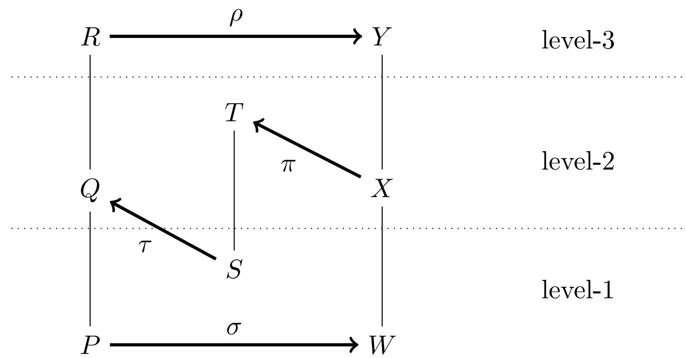
$\mathrm{rep}(R)$ is endowed with the $<_{BK}$ ordering

$$<^R = <_{BK} \restriction \mathrm{rep}(R).$$

R is Π^1_3-*wellfounded* iff

1. $\forall r \in \mathrm{dom}(R) \cup \{\emptyset\}$ $R\{r\}$ is Π^1_1-wellfounded, and
2. $\forall z \in [\mathrm{dom}(R)]$ $R(z) =_{\mathrm{DEF}} \cup_{n<\omega}(R_{\mathrm{tree}}(z \restriction n))_{1\leq n<\omega}$ is not Π^1_2-wellfounded.

For level-3 trees R and R', R is a *subtree* of R' iff R is a subfunction of R'. A *finite level-3 tower* is a sequence $(R_i)_{i\leq n}$ such that $n < \omega$, each R_i is a



regular level-2 tree, $\mathrm{card}(R_i) = i + 1$ and $i < j \to R_i$ is a subtree of R_j. $\vec{R}$ is *regular* iff each R_i is regular.

- W-description
- (P, W)-factoring map
- (Q, W)-description
- (S, Q, W)-factoring map
- (T, Q, W)-description
- (X, T, Q)-factoring map
- (Y, T, Q)-description
- (R, Y, T)-factoring map

Definition 0.18. Suppose W is a finite level-1 tree and suppose Q is a level ≤ 2 tree. A (Q, W)-*description* is of the form

$$\mathbf{D} = (d, (\mathbf{q}, \sigma))$$

such that either

1. $d = 1$, $\mathbf{q} \in {}^1Q$, $\sigma = \emptyset$, or
2. $d = 2$, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}({}^2Q)$, σ factors (P, W).

$\mathrm{desc}(Q, W)$ is the set of (Q, W)-descriptions. A $(Q, *)$-description is a (Q, W')-description for some finite level-1 tree W'. $\mathrm{desc}(Q, *)$ is the set of $(Q, *)$-descriptions. We sometimes abbreviate $(d, \mathbf{q}, \sigma)$ for $(d, (\mathbf{q}, \sigma)) \in \mathrm{desc}(Q, W)$ without confusion.

Suppose now $\mathbf{D} = (d, \mathbf{q}, \sigma)$ and if $d = 2$, then $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, $\mathrm{lh}(q) = k$. The *degree* of $\mathbf{D}$ is d. The *signature* of $\mathbf{D}$ is

$$\mathrm{sign}(\mathbf{D}) = \begin{cases} \emptyset & \text{if } d = 1, \\ (\sigma(p_i))_{i<k} & \text{if } d = 2. \end{cases}$$

$\mathbf{D}$ is *of continuous type* iff $d = 2$ and $\mathbf{q}$ is of continuous type; otherwise, $\mathbf{D}$

is *of discontinuous type.*

The *uniform cofinality* of $\mathbf{D}$ is

$$\mathrm{ucf}(\mathbf{D}) = \begin{cases} -1 & \text{if } d = 1 \vee (d = 2 \wedge \mathrm{ucf}(P, \vec{p}) = -1), \\ \sigma(\mathrm{ucf}(P, \vec{p})) & \text{if } d = 2 \wedge \mathrm{ucf}(P, \vec{p}) \neq -1. \end{cases}$$

The $*$-*signature* of $\mathbf{D}$ is

$$\mathrm{sign}_*(\mathbf{D}) = \begin{cases} ((1, \mathbf{q})) & \text{if } d = 1, \\ ((2, q \restriction i))_{1 \leq i \leq k-1} & \text{if } d = 2, q \text{ of continuous type}, \\ ((2, q \restriction i))_{1 \leq i \leq k} & \text{if } d = 2, q \text{ of discontinuous type}. \end{cases}$$

$\mathbf{D}$ is *of $*$-W-continuous type* iff either $d = 1$ or ($d = 2$ and ($\mathrm{ucf}(P, \vec{p}) \neq -1 \wedge \sigma(\mathrm{ucf}(P, \vec{p})) \neq \min(\prec^W)$) implies $\mathrm{pred}_{\prec^W}(\sigma(\mathrm{ucf}(P, \vec{p}))) \in \mathrm{ran}(\sigma)$). Otherwise, $\mathbf{D}$ is *of $*$-W-discontinuous type.*

The $*$-*W-uniform cofinality* of $\mathbf{D}$ is

$$\mathrm{ucf}^W_*(\mathbf{D})$$

defined as follows. If $d = 1$, then $\mathrm{ucf}^W_*(\mathbf{D}) = (1, \mathbf{q})$. If $d = 2$, q is of continuous type,

1. if $\mathbf{D}$ is of $*$-W-continuous type, then $\mathrm{ucf}_*^W(\mathbf{D}) = (2, (q^-, P \setminus \{p_{k-1}\}, \vec{p}))$;

2. if $\mathbf{D}$ is of $*$-W-discontinuous type, then $\mathrm{ucf}_*^W(\mathbf{D}) = (2, (q^-, P, \vec{p}))$.

If $d = 2$, q is of discontinuous type,

1. if $\mathbf{D}$ is of $*$-W-continuous type, then $\mathrm{ucf}_*^W(\mathbf{D}) = (2, \mathbf{q})$;

2. if $\mathbf{D}$ is of $*$-W-discontinuous type, then $\mathrm{ucf}_*^W(\mathbf{D}) = (2, (q, P \cup \{p_k\}, \vec{p})$.

The *constant* $(Q, *)$*-description* is $(2, (\emptyset, \emptyset, ((0))), \sigma_0)$ where σ_0 is the unique that factors $(\emptyset, *)$, i.e., $\sigma_0(\emptyset) = \emptyset$. If Q is a level-2 tree, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}(Q)$, $l \leq \mathrm{lh}(q)$, define

$$\mathbf{q} \restriction l = (q \restriction l, \{p_i : i < l\}, (p_i)_{i \leq l}).$$

which is a Q-description.

If $\mathbf{D} = (2, \mathbf{q}, \sigma) \in \mathrm{desc}(Q, *)$, $\mathbf{q} = (q, P, \vec{p})$, $l \leq \mathrm{lh}(q)$, define

$$\mathrm{lh}(\mathbf{D}) = \mathrm{lh}(\mathbf{q})$$

and

$$\mathbf{D} \restriction l = (2, \mathbf{q} \restriction l, \sigma \restriction \{p_i : i < l\})$$

which is a $(Q, *)$-description. Define

$$\mathbf{D} \lhd \mathbf{D}'$$

iff $\mathbf{D} = \mathbf{D}' \restriction l$ for some $l < \mathrm{lh}(\mathbf{D}')$. Define $\mathbf{D}^- = \mathbf{D} \restriction \mathrm{lh}(\mathbf{D}) - 1$. Define $\lhd^{Q,W} = \lhd \restriction \mathrm{desc}(Q, W)$.

Put

$$\langle \mathbf{D} \rangle = \begin{cases} (1, \mathbf{q}) & \text{if } d = 1, \\ (2, \sigma \oplus \mathbf{q}) & \text{if } d = 2. \end{cases}$$

Define

$$\mathbf{D} \prec \mathbf{D}'$$

iff $\langle \mathbf{D} \rangle <_{BK} \langle \mathbf{D}' \rangle$, the ordering on subcoordinates in $\omega^{<\omega}$ again according to $<_{BK}$. For example, the constant $(Q, *)$-description $\mathbf{D}_0$ is the $\prec$-maximum, and we have $\langle \mathbf{D}_0 \rangle = (2, \emptyset)$. When $1 \leq \mathrm{card}({}^1Q) < \aleph_0$, the $\prec$-least $(Q, *)$-description is $(1, q, \emptyset)$, where q is the $<_{BK}$-least node in ${}^1Q$. When $W \neq \emptyset$, the $\prec$-least (Q, W)-description of degree 2 is $\mathbf{D}_W = (2, ((-1), \{(0)\}, ((0))), \sigma_W$ where $\sigma_W((0)) =$the $<_{BK}$-least node in W, and we have $\langle \mathbf{D}_W \rangle = (2, (\sigma_W(1), -$ Define $\prec^{Q,W} = \prec \restriction \mathrm{desc}(Q, W)$. Then $[\![\mathbf{D}]\!]^{Q,W} < [\![\mathbf{D}']\!]^{Q,W}$ iff $\mathbf{D} \prec^{Q,W} \mathbf{D}'$;

$[\![\mathbf{D}]\!]^{Q,W} = [\![\mathbf{D}']\!]^{Q,W}$ iff $\mathbf{D} = \mathbf{D}'$.

Suppose W is a level-1 proper subtree of W', W' is finite, $w \in W \cup \{\emptyset\}$, $w' \in W' \setminus W$. Define

$$w = w' \restriction W$$

iff $w' <_{BK} w$ and $\{w^* \in W : w' <_{BK} w^* <_{BK} w\} = \emptyset$.

Suppose W is a proper level-1 subtree of W'. For $\mathbf{D} \in \mathrm{desc}(Q, W)$ and $\mathbf{D}' \in \mathrm{desc}(Q, W') \setminus \mathrm{desc}(Q, W)$, define

$$\mathbf{D} = \mathbf{D}' \restriction (Q, W)$$

iff $\mathbf{D}' \prec \mathbf{D}$ and $\{\mathbf{D}^* \in \mathrm{desc}(Q, W) : \mathbf{D}' \prec \mathbf{D}^* \prec \mathbf{D}\} = \emptyset$. Thus, $\mathbf{D} = \mathbf{D}' \restriction (Q, W)$ iff both $\mathbf{D}, \mathbf{D}'$ are of degree 2 and letting $\mathbf{D} = (2, (q, P, \vec{p}), \sigma)$, $\mathbf{D}' = (2, (q', P', \vec{p}'), \sigma')$, $\mathrm{lh}(q) = k$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, then either

1. q is of continuous type (hence $\mathrm{lh}(\vec{p}) = k$), $\mathbf{D}^- \lhd \mathbf{D}'$, $\sigma(p_{k-1}) = \sigma'(p_{k-1}) \restriction W$, or
2. q is of discontinuous type (hence $\mathrm{lh}(\vec{p}) = k+1$), $\mathbf{D} \lhd \mathbf{D}'$, $\sigma(p_k^-) = \sigma'(p_k) \restriction W$.

Suppose Q is a proper subtree of Q', both finite. For $(d, \mathbf{q}) \in \operatorname{desc}^*(Q)$, $(d', \mathbf{q}') \in \operatorname{desc}^*(Q') \setminus \operatorname{desc}^*(Q)$, define

$$(d, \mathbf{q}) = (d', \mathbf{q}') \upharpoonright Q$$

iff $(d', \mathbf{q}') \prec (d, \mathbf{q})$ and $\{(d^*, \mathbf{q}^*) \in \operatorname{desc}^*(Q) : (d', \mathbf{q}') \prec (d^*, \mathbf{q}^*) \prec (d, \mathbf{q})\} = \emptyset$. Thus, $(d, \mathbf{q}) = (d', \mathbf{q}') \upharpoonright Q$ iff either

1. $d = d' = 1$, $\mathbf{q} = \mathbf{q}' \upharpoonright^1 Q$, or
2. $d' = 1$, $\emptyset = \mathbf{q}' \upharpoonright^1 Q$, $d = 2$, $\mathbf{q} \in \{((-1), \{(0)\}, ((0))), (\emptyset, \emptyset, (0))\}$, or
3. $d = d' = 2$, letting $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\operatorname{lh}(\vec{p})}$, $\mathbf{q} = (q', P', \vec{p}')$, $\vec{p}' = (p'_i)_{i<\operatorname{lh}(\vec{p}')}$, $\operatorname{lh}(q) = k$, then either
   (a) $\mathbf{q} \in \operatorname{desc}(Q)$ is of continuous type, $k \geq 2$, $(P, \vec{p}) = (P', \vec{p}' \upharpoonright k)$, $(q^-)^- \subsetneq q'$, $q(k-2) = q'(k-2) \upharpoonright^2 Q\{(q^-)^-\}$, or
   (b) $\mathbf{q} \in \operatorname{desc}(Q)$ is of discontinuous type, $k \geq 2$, $(P, \vec{p} \upharpoonright k) = (P', \vec{p}' \upharpoonright k)$, $(q^-)^- = (q')^-$, $q(k-1) = q'(k-1) \upharpoonright^2 Q\{q^-\}$, or
   (c) $\mathbf{q} \notin \operatorname{desc}(Q)$, $q \subsetneq q'$, $\emptyset = q'(k) \upharpoonright^2 Q\{q^-\}$.

Suppose Q is a proper subtree of Q', both finite. For $\mathbf{D} \in \operatorname{desc}(Q, W)$,

$\mathbf{D}' \in \operatorname{desc}(Q', W) \setminus \operatorname{desc}(Q, W)$. Define

$$\mathbf{D} = \mathbf{D}' \restriction (Q, W)$$

iff $\mathbf{D}' \prec \mathbf{D}$ and $\{\mathbf{D}^* \in \operatorname{desc}(Q, W) : \mathbf{D}' \prec \mathbf{D}^* \prec \mathbf{D}\} = \emptyset$. Thus, putting $\mathbf{D} = (d, \mathbf{q}, \sigma)$, $\mathbf{D} = (d', \mathbf{q}', \sigma')$, $\mathbf{D} = \mathbf{D}' \restriction (Q, W)$ iff either

1. $d = d' = 1$, $\mathbf{q} = \mathbf{q}' \restriction^1 Q$, or
2. $d' = 1$, $\emptyset = \mathbf{q}' \restriction^1 Q$, $d = 2$, $\mathbf{q} = ((-1), \{(0)\}, ((0)))$, $\sigma((0)) = \min(\prec^W)$, or
3. $d = d' = 2$, letting $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\operatorname{lh}(\vec{p})}$, $\mathbf{q}' = (q', P', \vec{p}')$, $\vec{p}' = (p'_i)_{i<\operatorname{lh}(\vec{p}')}$, $\operatorname{lh}(q) = k$, then either
   (a) q is of continuous type (hence $\operatorname{lh}(\vec{p}) = k$), and either
      i. $\mathbf{D}$ is of $*$-W-discontinuous type, $\mathbf{D}^- \lhd \mathbf{D}'$, $\sigma'(p_{k-1}) = \operatorname{pred}_{\prec^W}(\sigma(p_{k-1}))$, $q'(k-1) \restriction^2 Q\{q^-\} = \emptyset$, or
      ii. $\mathbf{D}$ is of $*$-W-continuous type, $(\mathbf{D}^-)^- \lhd \mathbf{D}'$, $q'(k-2) \restriction^2 Q\{(q^-)^-\} = q(k-2)$, or
   (b) q is of continuous type (hence $\operatorname{lh}(\vec{p}) = k + 1$), and either

i. $\mathbf{D}$ is of $*$-W-discontinuous type, $\mathbf{D} \lhd \mathbf{D}'$, $\sigma'(p_k) = \mathrm{pred}_{\prec^W}(\sigma(p_k^-))$, $q'(k) \restriction^2 Q\{q\} = \emptyset$, or

ii. $\mathbf{D}$ is of $*$-W-continuous type, $\mathbf{D}^- \lhd \mathbf{D}'$, $q'(k-1) \restriction^2 Q\{q^-\} = q(k-1)$.

Suppose S is a finite regular level-1 tree and Q is a level ≤ 2 tree. Suppose $\tau : S \cup \{\emptyset\} \to \mathrm{desc}(Q, *)$ is a function. Then τ factors $(S, Q, *)$ iff

1. $\tau(\emptyset)$ is the constant $(Q, *)$-description.
2. If $s \prec^S s'$, then $\tau(s) \prec \tau(s')$.

For a level-1 tree W, τ factors (S, Q, W) iff τ factors $(S, Q, *)$ and $\mathrm{ran}(\tau) \subseteq \mathrm{desc}(Q, W)$. In particular, if every $\tau(s)$ is of degree 1, then τ factors $(S, Q, \emptyset)$.

If S is a level-1 tree, then

$$\mathrm{id}_{*,S}$$

factors (S, Q^0, S), where $\mathrm{id}_{*,S}(s) = (2, ((-1), \{(0)\}, ((0))), \sigma_s)$, $\sigma_s(0) = s$. It is convenient to label the nodes of a tree of uniform cofinalities using arbitrary sets instead of elements in $\omega^{<\omega}$ and $(\omega^{<\omega})^{<\omega}$. Suppose Q is a level ≤ 2 tree and W is a level-1 tree. *A representation of* $Q \otimes W$ is a pair (S, τ) such that

S is a level-1 tree, τ factors (S, Q, W), $\operatorname{ran}(\tau) = \operatorname{desc}(Q, W)$, and $s \prec^S s'$ iff $\tau(s) \prec^{Q,W} \tau(s')$. Representations of $Q \otimes W$ are clearly mutually isomorphic. We shall informally regard

$$Q \otimes W = \operatorname{desc}(Q, W) \setminus \{\text{the constant } (Q, W)\text{-description}\}$$

as a "level-1 tree" by identifying it with S via τ. If τ' factors (S', Q, W), then τ' also factors "level-1 trees" $(S', Q \otimes W)$, and $(\tau')^{Q \otimes W}$ makes sense. That is, $(\tau')^{Q \otimes W} = (\tau^{-1} \circ \tau')^S$, where $\tau^{-1} \circ \tau'$ factors (S', S). The identity map $\mathrm{id}_{Q \otimes W} : \mathbf{D} \mapsto \mathbf{D}$ factors $(Q \otimes W, Q, W)$. If Q is a subtree of Q' and W is a subtree of W', then $Q \otimes W$ is regarded as a subtree of $Q' \otimes W'$, and the map $j^{Q \otimes W, Q' \otimes W'}$ makes sense. In other words, let (S, τ) be a representation of $Q \otimes W$ and (S', τ') be a representation of $Q' \otimes W'$ such that S is a subtree of S' and $\tau \subseteq \tau'$, then $j^{Q \otimes W, Q' \otimes W'} = j^{S, S'}$.

If π factors level ≤ 2 trees (Q, T), then

$$\pi \otimes W$$

factors level-1 trees $(Q \otimes W, T \otimes W)$, where $\pi(d, \mathbf{q}, \sigma) = (d, {}^d\pi(\mathbf{q}), \sigma)$.

Definition 0.19. Suppose $I < \omega$. Suppose for each $i < I$, $\bar{J}_i \leq J_i < \omega$ and $A_i = (a_{i,j})_{\bar{J}_i \leq j < J_i}$ is a finite sequence of sets. Then the *contraction* of $(A_i)_{i<I}$ is $(b_k)_{k<K}$ such that

1. $\{a_{i,j} : i < I, \bar{J}_i \leq j < J_i\} = \{b_k : k < K\}$.
2. For each $k < K$, letting (i^k, j^k) be the $<_{BK}$-least (i,j) such that $a_{i,j} = b_k$, then the map $k \mapsto (i^k, j^k)$ is order preserving with respect to $<$ and $<_{BK}$.

Definition 0.20. Suppose T, Q are level ≤ 2 trees. A $(T, Q, -1)$-*description* is just a $(T, {}^1Q)$-description. Suppose $(W, \vec{w})$ is a potential partial level ≤ 1 tower of discontinuous type, $\vec{w} = (w_i)_{i \leq m}$. If $m = 0$, the only $(T, Q, (W, \vec{w}))$-*description* is $(2, ((\emptyset, \emptyset, ((0))), \tau))$, where τ factors $(\emptyset, Q, \emptyset)$, which is called *the constant* $(T, Q, *)$-*description*. If $m > 0$, a $(T, Q, (W, \vec{w}))$-*description* is of the form

$$\mathbf{C} = (2, (\mathbf{t}, \tau))$$

such that

1. $\mathbf{t} \in \operatorname{desc}({}^2T)$ and $\mathbf{t} \neq (\emptyset, \emptyset, (0))$. Let $\mathbf{t} = (t, S, \vec{s})$, $\operatorname{lh}(t) = k$, $\vec{s} =$

$(s_i)_{i<\mathrm{lh}(\vec{s})}$.

2. τ factors (S, Q, W).
3. The contraction of $(\mathrm{sign}(\tau(s_i)))_{i<k}$ is $(w_i)_{i<m}$.
4. If t is of continuous type and w_{m-1} does not appear in the contraction of $(\mathrm{sign}(\tau(s_i)))_{i<k-1}{}^\frown(\mathrm{sign}(\tau(s_{k-1})^-))$ then $\tau(s_{k-1})$ is of discontinuous type.
5. Either $\mathrm{ucf}(S, \vec{s}) = w_m = -1$ or $\mathrm{ucf}_1(\tau(\mathrm{ucf}(S, \vec{s}))) = \mathrm{ucf}(W, \vec{w})$.

A $(T, Q, *)$-description is either a $(T, Q, -1)$-description or a $(T, Q, (W', \vec{w}'))$-description for some potential partial level ≤ 1 tower $(W', \vec{w}')$ of discontinuous type. For a level-1 tree W, a (T, Q, W)-description is a $(T, Q, (W, \vec{w}'))$-description for some $\vec{w}'$. $\mathrm{desc}(T, Q, -1)$, $\mathrm{desc}(T, Q, (W, \vec{w}))$, $\mathrm{desc}(T, Q, *)$, $\mathrm{desc}(T, Q, W)$ denote the sets of relevant descriptions. We sometimes abbreviate $(d, \mathbf{t}, \tau)$ for $(d, (\mathbf{t}, \tau)) \in \mathrm{desc}(T, Q, *)$ without confusion. Recalling our notation of $Q \otimes W$, we may regard $\mathrm{desc}(T, Q, -1) \subseteq \mathrm{desc}(T, \emptyset)$, $\mathrm{desc}(T, Q, W) \subseteq \mathrm{desc}(T, Q \otimes W)$. $T \otimes (Q \otimes W)$ is also a "level-1 tree", whose nodes consist of non-constant $(T, Q \otimes W)$-descriptions, so that $\operatorname{desc}(T, Q \otimes W) = \operatorname{desc}(T \otimes (Q \otimes W))$.

Every non-constant (T, Q, W)-description is a member of $T \otimes (Q \otimes W)$. The constant $(T, Q, *)$-description $\mathbf{C}_0$ is regarded as the constant $T \otimes (Q \otimes W)$-description, to make sense of $\operatorname{seed}^{T \otimes (Q \otimes W)}_{\mathbf{C}_0}$. The *degree* of $(d, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, *)$ is d. In fact, if $\mathbf{C} \in \operatorname{desc}(T, Q, *)$ is of degree 2, then there is a unique potential partial level ≤ 1 tower $(W, \vec{w})$ for which $\mathbf{C} \in \operatorname{desc}(T, Q, (W, \vec{w}))$.

Suppose now $\mathbf{C} = (d, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, *)$, and if $d = 2$, then $\mathbf{C} \in \operatorname{desc}(T, Q, (W, \vec{w}))$, $\mathbf{t} = (t, S, \vec{s})$, $\operatorname{lh}(t) = k$, $\vec{s} = (s_i)_{i < \operatorname{lh}(\vec{s})}$, $\vec{w} = (w_i)_{i \leq m}$. If $d = 1$ and the signature of $\mathbf{C}$ regarded as a $(T, {}^1Q)$ is $(q_i)_{i<l}$, then the *signature* of $\mathbf{C}$ is $((1, q_i))_{i<l}$. If $d = 2$, the *signature* of $\mathbf{C}$ is

$$\operatorname{sign}(\mathbf{C}) = \text{the contraction of } (\operatorname{sign}^W_*(\tau(s_i)))_{i<k}.$$

$\mathbf{C}$ is *of continuous type* iff $d = 2$, t is of continuous type, and $\tau(s_{k-1})$ is of $*$-W-continuous type; $\mathbf{C}$ is *of discontinuous type* otherwise. The *uniform cofinality* of $\mathbf{C} = (d, \mathbf{t}, \tau)$ is

$$\operatorname{ucf}(\mathbf{C})$$

defined as follows. If $d = 1$ and $\mathrm{ucf}_{\mathbf{C}} = q_*$ regarding $\mathbf{C}$ as a $(T, {}^1Q)$-description, then $\mathrm{ucf}(\mathbf{C}) = (0, -1)$ when $q_* = -1$, $\mathrm{ucf}(\mathbf{C}) = (1, q_*)$ otherwise. If $d = 2$ then

1. if $\mathrm{ucf}(S, \vec{s}) = -1$, then $\mathrm{ucf}(\mathbf{C}) = (0, -1)$;
2. if $\mathrm{ucf}(S, \vec{s}) = s_* \neq -1$, then $\mathrm{ucf}(\mathbf{C}) = \mathrm{ucf}_*^W(\tau(s_*))$.

If $w_m \neq -1$, $\mathbf{C}$ is said to be of *plus-discontinuous type*, and put

$$\mathrm{ucf}^+(\mathbf{C}) = \mathrm{ucf}_*^{W^+}(\tau(\mathrm{ucf}(S, \vec{s}))),$$

where W^+ is the completion of (W, w_m). The $*$-*signature* of $\mathbf{C}$ is

$$\mathrm{sign}_*(\mathbf{C}) = \begin{cases} ((1, \mathbf{t})) & \text{if } d = 1, \\ ((2, t \restriction i))_{1 \leq i \leq k-1} & \text{if } d = 2, t \text{ is of continuous type}, \\ ((2, t \restriction i))_{1 \leq i \leq k} & \text{if } d = 2, t \text{ is of discontinuous type}. \end{cases}$$

If $d = 1$, $\mathbf{C}$ is *of $*$-Q-continuous type* iff $\mathbf{C}$ is of $*$-${}^1Q$-continuous type regarded as a $(T, {}^1Q)$-description. If $d = 2$, $\mathbf{C}$ is *of $*$-Q-continuous type* iff $\mathrm{ucf}(S, \vec{s}) \neq -1 \wedge \tau(\mathrm{ucf}(S, \vec{s})) \neq \min(\prec^{Q,W}))$ implies $\mathrm{pred}_{\prec^{Q,W}}(\tau(\mathrm{ucf}(S, \vec{s}))) \in \mathrm{ran}(\tau)$. $\mathbf{C}$ is *of $*$-Q-discontinuous type* iff $\mathbf{C}$ is not of $*$-Q-continuous type.

The $*$-Q-*uniform cofinality* of $\mathbf{C}$ is

$$\mathrm{ucf}^Q_*(\mathbf{C})$$

defined as follows. If $d = 1$, then $\mathrm{ucf}^Q_*(\mathbf{C})$ is defined just by treating $\mathbf{C}$ as a $(T, {}^1Q)$-description. If $d = 2$, t is of continuous type,

1. if $\mathbf{C}$ is of $*$-Q-continuous type, then $\mathrm{ucf}^Q_*(\mathbf{C}) = (2, (t^-, S \setminus \{s_{k-1}\}, \vec{s}))$;
2. if $\mathbf{C}$ is of $*$-Q-discontinuous type, then $\mathrm{ucf}^Q_*(\mathbf{C}) = (2, (t^-, S, \vec{s}))$.

If $d = 2$, t is of discontinuous type,

1. if $\mathbf{C}$ is of $*$-Q-continuous type, then $\mathrm{ucf}^Q_*(\mathbf{C}) = (2, \mathbf{t})$;
2. if $\mathbf{C}$ is of $*$-Q-discontinuous type, then $\mathrm{ucf}^Q_*(\mathbf{C}) = (2, (t, S \cup \{s_k\}, \vec{s}))$.

If $\mathbf{C} = (2, \mathbf{t}, \tau)$, $\mathbf{t} = (t, S, \vec{s})$, define

$$\mathrm{lh}(\mathbf{C}) = \mathrm{lh}(\mathbf{t})$$

and

$$\mathbf{C} \restriction l = (2, \mathbf{t} \restriction l, \tau \restriction \{s_i : i < l\}).$$

Define

$$\mathbf{C} \lhd \mathbf{C}'$$

iff $\mathbf{C} = \mathbf{C}' \restriction l$ for some $l < \mathrm{lh}(\mathbf{C}')$. Define $\mathbf{C}^- = \mathbf{C} \restriction \mathrm{lh}(\mathbf{C}) - 1$

Let

$$\langle \mathbf{C} \rangle = \begin{cases} (1, \mathbf{t}) & \text{if } d = 1, \\ (2, \tau \oplus \mathbf{t}) & \text{if } d = 2. \end{cases}$$

Define

$$\mathbf{C} \prec \mathbf{C}'$$

iff $\langle \mathbf{C} \rangle <_{BK} \langle \mathbf{C}' \rangle$, the ordering on subcoordinates in $\mathrm{desc}(Q, *) \cup \mathrm{desc}(Q', *)$ according to $\prec$ acting on $\mathrm{desc}(Q, *) \cup \mathrm{desc}(Q', *)$.

The constant $(T, Q, *)$-description, $\mathbf{C}_0$, is the $\prec$-greatest, and we have $\langle \mathbf{C}_0 \rangle = (2, \emptyset)$. Define $\prec^{T,Q} = \prec \restriction \mathrm{desc}(T, Q, *)$. Suppose $(\vec{W}, \vec{w}) = (W_i, w_i)_{i \leq m}$ is a potential partial level-1 tower and $m > 0$. If $\mathbf{C} = (2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (W_m$ $\mathbf{t} = (t, S, \vec{s})$, $\bar{m} < m$, then

$$\mathbf{C} \restriction (T, Q, W_{\bar{m}}) \in \mathrm{desc}(T, Q, (W_{\bar{m}}, (w_i)_{i \leq \bar{m}}))$$

is defined by the following: letting l be the least such that $\tau(s_l) \notin \mathrm{desc}(Q, W_{\bar{m}})$, and letting $\mathbf{D} = \tau(s_l) \restriction (Q, W_{\bar{m}})$, then

1. if $\mathbf{D} \neq \tau(s_l^-)$, then $\mathbf{C} \restriction (T, Q, W_{\bar{m}}) = (2, \mathbf{t} \restriction l^\frown(-1), \bar{\tau})$, where $\bar{\tau}$ and τ

agree on ${}^2T_{\rm tree}(t \restriction l)$, $\bar{\tau}(s_l) = \mathbf{D}$;

2. if $\mathbf{D} = \tau(s_l^-)$, then $\mathbf{C} \restriction (T, Q, W_{\bar{m}}) = (2, \mathbf{t} \restriction l, \tau \restriction {}^2T_{\rm tree}(t \restriction l))$.

Suppose X, T, Q are level ≤ 2 trees. Suppose $\pi : \mathrm{dom}(X) \to \mathrm{desc}(T, Q, *)$ is a function. π is said to *factor* (X, T, Q) iff

1. If $(1, x) \in \mathrm{dom}(X)$, then $\pi(1, x) \in \mathrm{desc}(T, Q, -1) \cup \mathrm{desc}(T, Q, {}^2X[\emptyset])$.
2. If $(2, x) \in \mathrm{dom}(X)$, then $\pi(2, x) \in \mathrm{desc}(T, Q, {}^2X[x])$.
3. $\pi(2, \emptyset)$ is the constant $(T, Q, *)$-description.
4. For any $(d, x), (d', x') \in \mathrm{dom}(X)$, if $(d, x) <_{BK} (d', x')$ then $\pi(d, x) \prec^{T,Q} \pi(d', x')$.
5. For any $x \in \mathrm{dom}({}^2X) \setminus \{\emptyset\}$, $\pi(2, x^-) = \pi(2, x) \restriction (T, Q, {}^2X_{\rm tree}(x^-))$.

π is said to *factor* $(X, T, *)$ iff π factors (X, T, Q') for some level ≤ 2 tree Q'.

Suppose T, Q are both level ≤ 2 trees. *A representation of* $T \otimes Q$ is a pair (X, π) such that

1. X is a level ≤ 2 tree;

2. π factors (X, T, Q);

3. $\mathrm{ran}(\pi) = \mathrm{desc}(T, Q, *)$;

4. $(d, x) <_{BK} (d', x')$ iff $\pi(d, x) \prec^{T,Q} \pi(d', x')$.

Representations of $T \otimes Q$ are clearly mutually isomorphic. We shall regard

$$T \otimes Q$$

itself as a "level ≤ 2 tree" whose level-d component is $(\mathbf{t}, \tau)$ for which $(d, (\mathbf{t}, \tau)) \in \mathrm{desc}(T, Q, *)$, and whose level-2 component sends $(\mathbf{t}, \tau)$ to (W, w_m) if $(2, (\mathbf{t}, \tau)) \in \mathrm{desc}(T, Q, (W, (w_i)_{i \leq m}))$. In this way, π is a "level ≤ 2 tree isomorphism" between X and $T \otimes Q$. All the relevant terminologies of level ≤ 2 trees carry over to $T \otimes Q$ in the obvious ways.

In particular, if W is a finite level-1 tree, a $(T \otimes Q, W)$-description takes one of the following forms (recall that $(d, \mathbf{t}, \tau)$ is simply an abbreviation of $(d, (\mathbf{t}, \tau))$):

1. $(1, (\mathbf{t}, \emptyset), \emptyset)$ for $(1, \mathbf{t}, \emptyset) \in \mathrm{desc}(T, Q, -1)$;

2. $(2, ((\mathbf{t}, \tau), Z, \vec{z}), \psi)$ for $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$ and ψ factoring (Z, W);

3. $(2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi)$ for $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$, $\vec{z} = (z_i)_{i \leq l}$, $Z^+ = Z \cup \{z_l\}$ and ψ factoring (Z^+, W).

$(T \otimes Q) \otimes W$ is thus regarded as a "level-1 tree" whose nodes consists of non-constant $(T \otimes Q, W)$-descriptions.

There is a natural isomorphism

$$\iota_{T,Q,W}$$

between "level-1 trees" $(T \otimes Q) \otimes W$ and $T \otimes (Q \otimes W)$, defined as follows.

1. $\iota_{T,Q,W}(1, (\mathbf{t}, \emptyset), \emptyset) = (1, \mathbf{t}, \emptyset)$.
2. If $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$, $\mathbf{t} = (t, S, \vec{s})$, ψ factors (Z, W), define $\iota_{T,Q,W}(2, ((\mathbf{t}, \tau), Z, \vec{z}), \psi) = (2, \mathbf{t}, (Q \otimes \psi) \circ \tau)$.
3. If $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$, $\mathbf{t} = (t, S, \vec{s})$, $\vec{z} = (z_i)_{i \leq l}$, $\vec{s} = (s_i)_{i \leq k}$, $Z^+ = Z \cup \{z_l\}$, ψ factors (Z^+, W),
   (a) if t is of discontinuous type, define $\iota_{T,Q,W}(2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi) = (2, \mathbf{t}^\frown(-1), \psi *_0 \tau)$, where $\psi *_0 \tau$ factors $(S \cup \{s_k\}, Q, W)$, $\psi *_0 \tau$ extends $(Q \otimes \psi) \circ \tau$, $\psi *_0 \tau(s_k) = (2, \mathbf{q}_0, \sigma)$, $\mathbf{q}_0 = ((-1), \{(0)\}, ((0)))$, $\sigma((0)) =$

$\psi(z_l)$;

(b) if t is of continuous type, define $\iota_{T,Q,W}(2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi) = (2, \mathbf{t}, \psi *_1 \tau)$, where $\psi *_1 \tau$ factors (S, Q, W), $\psi *_1 \tau$ extends $(Q \otimes \psi) \circ (\tau \restriction (S \setminus \{s_k\}))$, $\psi *_1 \tau(s_k) = (2, \mathbf{q}^\frown(-1), \sigma^+)$ where $\tau(s_k) = (2, \mathbf{q}, \sigma)$, $\mathbf{q} = (q, P, (p_i)_{i \leq m})$, σ^+ extends σ, $\sigma^+(p_m) = \psi(z_l)$.

$\iota_{T,Q,W}$ justifies the associativity of the $\otimes$ operator acting on level $(\leq 2, \leq 2, 1)$-trees.

Suppose T, Q, U are level ≤ 2 trees. There is a natural "level ≤ 2 tree isomorphism"

$$\iota_{T,Q,U}$$

between $(T \otimes Q) \otimes U$ and $T \otimes (Q \otimes U)$ defined as follows. Suppose $\mathbf{B} \in \text{desc}(T \otimes Q, U, *)$.

1. If $\mathbf{B} = (1, (t, \emptyset), \emptyset)$, $\mathbf{C} = (1, t, \emptyset) \in \text{desc}(T, Q, -1)$, then $\mathbf{C} \in \text{desc}(T, Q \otimes U, -1)$ and $\iota_{T,Q,U}(\mathbf{B}) = \mathbf{C}$.
2. If $\mathbf{B} = (2, ((\mathbf{t}, \tau), Z, \vec{z}), \psi) \in \text{desc}(T \otimes Q, U, W)$, $\mathbf{C} = (2, \mathbf{t}, \tau) \in \text{desc}(T, Q, ($

$\mathbf{t} = (t, S, \vec{s})$, ψ factors (Z, U, W), then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}, \iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ \tau)$.

3. If $\mathbf{B} = (2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi) \in \operatorname{desc}(T \otimes Q, U, W)$, $\mathbf{C} = (2, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, (Z, \vec{z}))$, $\mathbf{t} = (t, S, (s_i)_{i \leq k})$, $\vec{z} = (z_i)_{i \leq l}$,
   (a) if t is of discontinuous type, then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}^\frown(-1), \psi *_0 \tau)$, where $\psi *_0 \tau$ factors $(S \cup \{s_k\}, Q \otimes U, W)$, $\psi *_0 \tau$ extends $\iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ \tau$, $\psi *_0 \tau(s_k) = \iota^{-1}_{Q,U,W}(2, \mathbf{q}_0, \sigma)$, $\mathbf{q}_0 = ((-1), \{(0)\}, ((0)))$, $\sigma((0)) = \psi(z_l)$.
   (b) if t is of continuous type, then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}, \psi *_1 \tau)$, where $\psi *_1 \tau$ factors $(S, Q \otimes U, W)$, $\psi *_1 \tau$ extends $\iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ (\tau \restriction (S \setminus \{s_k\}))$, $\psi *_1 \tau(s_k) = \iota^{-1}_{Q,U,W}(2, \mathbf{q}^\frown(-1), \sigma^+)$ where $\tau(s_k) = (2, \mathbf{q}, \sigma)$, $\mathbf{q} = (q, P, (p_i)_{i \leq m})$, σ^+ extends σ, $\sigma^+(p_m) = \psi(z_l)$.

$\iota_{T,Q,U}$ justifies the associativity of the $\otimes$ operator acting on level $(\leq 2, \leq 2, \leq 2)$ trees.

Suppose Q, Q' are finite level ≤ 2 trees, Q is a proper subtree of Q', $(W_i, w_i)_{i \leq m'}$ is a partial level ≤ 1 tower, $m \leq m'$, $\mathbf{C} \in \operatorname{desc}(T, Q, (W_m, (w_i)_{i \leq

$\mathbf{C}' \in \mathrm{desc}(T, Q', (W_{m'}, (w_i)_{i \leq m'})) \setminus \mathrm{desc}(T, Q, (W_m, (w_i)_{i \leq m}))$. Define

$$\mathbf{C} = \mathbf{C}' \restriction (T, Q)$$

iff $\mathbf{C}' \prec \mathbf{C}$ and $\bigcup_{m \leq k \leq m'} \{\mathbf{C}^* \in \mathrm{desc}(T, Q, (W_k, (w_i)_{i \leq k})) : \mathbf{C}' \prec \mathbf{C}^* \prec \mathbf{C}\} = \emptyset$. A purely combinatorial argument shows that $\mathbf{C} = \mathbf{C}' \restriction (T, Q)$ iff $\mathbf{C}$ and $\mathbf{C}'$ are both of degree 2 and putting $\mathbf{C} = (2, \mathbf{t}, \tau)$, $\mathbf{C}' = (2, \mathbf{t}', \tau')$, $\mathbf{t} = (t, S, \vec{s})$, $\vec{s} = (s_i)_{i < \mathrm{lh}(\vec{s})}$, $\mathbf{t}' = (t', S', \vec{s}')$, $k = \mathrm{lh}(t)$, then either

1. t is of continuous type, $\mathbf{C}^- \lhd \mathbf{C}$, $\tau(s_{k-1}) = \tau'(s_{k-1}) \restriction (Q, W_{m'})$, or
2. t is of discontinuous type, $\mathbf{C} \lhd \mathbf{C}'$, $\tau(s_k^-) = \tau'(s_k) \restriction (Q, W_{m'})$.

Note that if π factors Π^1_2-wellfounded trees (X, T), then $[\![d, x]\!]_X \leq [\![\pi(d, x)]\!]_T$ for any $(d, x) \in \mathrm{dom}(X)$. We say that π *minimally factors* (X, T) iff π factors (X, T), X, T are both Π^1_2-wellfounded and $[\![d, x]\!]_X = [\![\pi(d, x)]\!]_T$ for any $(d, x) \in \mathrm{dom}(X)$. In particular, if T, Q are both Π^1_2-wellfounded, then $\mathrm{id}_{T,*}$ minimally factors $(T, T \otimes Q)$.

A comparison theorem between Π^1_2-wellfounded trees:

Theorem 0.21. *Suppose X, T are Π^1_2-wellfounded level ≤ 2 trees. Then* *there exists* (Q, π) *such that* Q *is* Π^1_2*-wellfounded and* π *minimally factors* $(X, T \otimes Q)$.

Suppose R is a level-3 tree. The *constant* R*-description* is $\emptyset$. An R*-description* is either the constant R-description or a triple $(r, Q, \overrightarrow{(d, q, P)})$ such that either $r \in \operatorname{dom}(R) \wedge (Q, \overrightarrow{(d, q, P)}) = R[r]$ or $r = r^- {}^\frown (-1) \wedge r^- \in \operatorname{dom}(R) \wedge Q$ is a completion of $R(r^-) \wedge (Q, \overrightarrow{(d, q, P)}) = R[r, Q]$. $\operatorname{desc}(R)$ is the set of R-descriptions. $(r, Q, \overrightarrow{(d, q, P)})$ is *of discontinuous type* if $r \in \operatorname{dom}(R)$, *of continuous type* otherwise. If $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)})$ is of discontinuous type and Q^+ is a completion of $(Q, \overrightarrow{(d, q, P)})$, then

$$\mathbf{r}^\frown(-1, Q^+) = (r^\frown(-1), Q^+, \overrightarrow{(d, q, P)}).$$

An *extended* R*-description* is either an R-description or a triple $(r, Q, \overrightarrow{(d, q, P}$ such that $(r^\frown(-1), Q, \overrightarrow{(d, q, P)})$ is an R-description of continuous type. $\operatorname{desc}^*(R$ is the set of extended R-descriptions. An extended R-description $\mathbf{r}$ is *regular* iff either $\mathbf{r} \in \operatorname{desc}(R)$ of discontinuous type or $\mathbf{r} \notin \operatorname{desc}(R)$. A *generalized* R*-description* is either $(\emptyset, \emptyset, \emptyset)$ or of the form

$$\mathbf{A} = (\mathbf{r}, \pi, T)$$

so that $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)}) \in \operatorname{desc}(R) \setminus \{\emptyset\}$, T is a finite level ≤ 2 tree, π factors (Q, T). $\operatorname{desc}^{**}(R)$ is the set of generalized R-descriptions.

Suppose $(Q, (d, q, P))$ is a partial level ≤ 2 tree. We define

$$\operatorname{ucf}^*(Q, (d, q, P)) = \begin{cases} (0, -1, \emptyset) & \text{if } \operatorname{ucf}(Q, (d, q, P)) = (0, -1), \\ (1, q^*, \emptyset) & \text{if } \operatorname{ucf}(Q, (d, q, P)) = (1, q^*), \\ (2, \mathbf{q}^*, \operatorname{id}_{{}^2 Q_{\text{tree}}(q^*)}) & \text{if } \operatorname{ucf}(Q, (d, q, P)) = (2, \mathbf{q}^*), \\ & \quad \mathbf{q}^* = (q^*, P^*, \vec{p}^*). \end{cases}$$

Thus, $\operatorname{ucf}^*(Q, (d, q, P)) \in \{(0, -1, \emptyset)\} \cup \operatorname{desc}(Q, P)$, and $\operatorname{cf}(Q, (d, q, P)) = 1$ iff $\operatorname{ucf}^*(Q, (d, q, P)) = \min(\prec^{Q,P})$. If $\operatorname{cf}(Q, (d, q, P)) = 2$, let

$$\operatorname{ucf}^-(Q, (d, q, P)) = \operatorname{pred}_{\prec^{Q,P}}(\operatorname{ucf}^*(Q, (d, q, P))).$$

Put $\pi \oplus \emptyset = \emptyset$. Suppose Y is a level-3 tree, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)}) \in \operatorname{desc}(Y)$, $\operatorname{lh}(y) = k$, $\overrightarrow{(e, x, W)} = (e_i, x_i, W_i)_{1 \leq i \leq \operatorname{lh}(\vec{y})}$, π is a function whose domain contains $\operatorname{dom}(X)$, we put

$$\pi \oplus \mathbf{y} = \pi \oplus_Y y = (y(0), \pi(e_1, x_1), y(1), \dots, \pi(e_{k-1}, x_{k-1}), y(k-1)).$$

If $l < \operatorname{lh}(y)$, then $\mathbf{y} \restriction l = (y \restriction l, Y_{\text{tree}}(y \restriction l), (e_i, x_i, W_i)_{1 \leq i \leq l})$.

Suppose Y is a level-3 tree, T is a level ≤ 2 tree. The only $(Y, T, \emptyset)$-*description* is $(\emptyset, \emptyset)$, which is called the *constant* $(Y, T, *)$-*description*. Suppose $(Q, \overrightarrow{(d, q, P)}) = (Q, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a potential partial level ≤ 2 tower of discontinuous type. A $(Y, T, (Q, \overrightarrow{(d, q, P)}))$-*description* is of the form

$$\mathbf{B} = (\mathbf{y}, \pi)$$

with the following properties:

1. $\mathbf{y} \in \operatorname{desc}(Y) \setminus \{\emptyset\}$. Put $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)})$, $\operatorname{lh}(y) = l$, $\overrightarrow{(e, x, W)} = (e_i, x_i, W_i)_{1 \leq i \leq \operatorname{lh}(\vec{x})}$.
2. π factors (X, T, Q).
3. The contraction of $(\operatorname{sign}(\pi(e_i, x_i)))_{1 \leq i < l}$ is $((d_i, q_i))_{1 \leq i < k}$.
4. If $k > 1$, y is of continuous type and (d_{k-1}, q_{k-1}) does not appear in the contraction of $(\operatorname{sign}(\pi(e_i, x_i)))_{1 \leq i < l} {}^\frown (\operatorname{sign}(\pi(e_{l-1}, x_{l-1})^-))$, then $\pi(e_{l-1}, x_l$ is of discontinuous type.
5. Put $\operatorname{ucf}(X, \overrightarrow{(e, x, W)}) = (e_*, \mathbf{x}_*)$.

(a) If $e_* = 0$ then $d_k = 0$.

(b) If $e_* = 1$ then $\mathrm{ucf}(\pi(1, \mathbf{x}_*)) = \mathrm{ucf}(Q, \overrightarrow{(d, q, P)})$.

(c) If $e_* = 2$, $\mathbf{x}_* = (x_*, W_*, \vec{w}_*) \in \mathrm{desc}(X)$, then $\mathrm{ucf}(\pi(2, x_*)) = \mathrm{ucf}(Q, \overline{(d, q,}$

(d) If $e_* = 2$, $\mathbf{x}_* = (x_*, W_*, \vec{w}_*) \notin \mathrm{desc}(X)$, then $\mathrm{ucf}^+(\pi(2, x_*)) = \mathrm{ucf}(Q, \overline{(d,}$

A (Y, T, Q)-description is a $(Y, T, (Q, \overrightarrow{(d', q', P')}))$-description for some potential partial level ≤ 2 tower $(Q, \overrightarrow{(d', q', P')})$ of discontinuous type. A $(Y, T, *)$-description is a (Y, T, Q')-description for some level ≤ 2 tree Q' or $Q' = \emptyset$. $\mathrm{desc}(Y, T, (Q, \overrightarrow{(d, q, P)}))$, $\mathrm{desc}(Y, T, Q)$, $\mathrm{desc}(Y, T, *)$ denote the sets of relevant descriptions.

Given a $(Y, T, *)$-description $\mathbf{B} = (\mathbf{y}, \pi)$, define

$$\langle \mathbf{B} \rangle = \pi \oplus \mathbf{y}.$$

Define

$$\mathbf{B} \prec \mathbf{B}'$$

iff $\langle \mathbf{B} \rangle <_{BK} \langle \mathbf{B}' \rangle$, the ordering on coordinates in $\mathrm{desc}(T, Q, *)$ for some T, Q again according to $\prec$. The constant $(Y, T, *)$-description $\mathbf{B}_0$ is the $\prec$-greatest, and we have $\langle \mathbf{B}_0 \rangle = \emptyset$. Define $\prec^{Y,T} = \prec \restriction \mathrm{desc}(Y, T, *)$.

Suppose $(\vec{Q}, \overrightarrow{(d,q,P)}) = (Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a potential partial level ≤ 2 tower and $\mathbf{B} = (\mathbf{y}, \pi) \in \operatorname{desc}(Y, T, (Q_k, \overrightarrow{(d,q,P)}))$. Define

$$\mathbf{B} \restriction (Y, T, \emptyset) = \text{the constant } (Y, T, *)\text{-description.}$$

Suppose $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)})$, $0 < \bar{k} < k$. Then

$$\mathbf{B} \restriction (Y, T, Q_{\bar{k}}) \in \operatorname{desc}(Y, T, (Q_{\bar{k}}, (d_i, q_i, P_i)_{1 \leq i \leq \bar{k}}))$$

is defined by the following: letting l be the least such that $\pi(e_l, x_l) \notin \operatorname{desc}(T, Q_{\bar{k}}$ $\mathbf{C} \in \operatorname{desc}(T, Q_{\bar{k}}, *)$ be such that $\mathbf{C} = \pi(e_l, x_l) \restriction (T, Q_{\bar{k}})$, letting $(e_*, \mathbf{x}_*) = \operatorname{ucf}(Y[y \restriction l])$, $\mathbf{x}_* = x_*$ if $e_* = 1$, $\mathbf{x}_* = (x_*, \dots)$ if $e_* = 2$, then

1. if $\mathbf{C} \neq \pi(e_*, x_*)$, then $\mathbf{B} \restriction (Y, T, Q_{\bar{k}}) = (\mathbf{y} \restriction l^\frown(-1, X), \bar{\pi})$, where $\bar{\pi}$ and π agree on $Y_{\text{tree}}(y \restriction l)$, $\bar{\pi}(e_l, x_l) = \mathbf{C}$, and $\bar{\pi}$ factors $(X, T, Q_{\bar{k}})$;
2. if $\mathbf{C} = \pi(e_*, x_*)$, then $\mathbf{B} \restriction (Y, T, Q_{\bar{k}}) = (\mathbf{y} \restriction l, \pi \restriction Y_{\text{tree}}(y \restriction l))$.

Suppose R, Y are level-3 trees, T is a level ≤ 2 tree. Suppose $\rho : \operatorname{dom}(R) \cup \{\emptyset\} \to \operatorname{desc}(Y, T, *)$ is a function. ρ *factors* (R, Y, T) iff

1. $\rho(\emptyset)$ is the constant $(Y, T, *)$-description.

2. For any $r \in \mathrm{dom}(R)$, $\rho(r) \in \mathrm{desc}(Y, T, R[r])$.

3. For any $r^\frown(a), r^\frown(b) \in \mathrm{dom}(R)$, if $a <_{BK} b$ and $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown($ then $\rho(r^\frown(a)) \prec \rho(r^\frown(b))$.

4. For any $r \in \mathrm{dom}(R)$, $\rho(r^-) = \rho(r) \restriction (Y, T, R_{\mathrm{tree}}(r^-))$.

If Y is a level-3 tree, then

$$\mathrm{id}_{Y,*}$$

factors (Y, Y, Q^0) where $\mathrm{id}_{Y,*}(y) = ((y, X, \overrightarrow{(e, x, W)}), \mathrm{id}_{*,X})$ for $Y[y] = (X, \overrightarrow{(e, x, W)})$.

For level-3 trees R, Y, we say that $\rho : \mathrm{dom}(R) \to \mathrm{dom}(Y)$ *factors* (R, Y) iff

1. If $r \in \mathrm{dom}(R)$ then $R(r) = Y(\rho(r))$.

2. If $r, r' \in \mathrm{dom}(R)$ and $r \subseteq r'$, then $\rho(r) \subseteq \rho(r')$.

3. If $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown(b))$ and $a <_{BK} b$, then $\rho(r^\frown(a)) <_{BK} \rho(r^\frown(b))$.

If in addition, ρ is onto $\mathrm{dom}(Y)$, then ρ is called a *level-3 tree isomorphism* between R and Y. Suppose Y is a level-3 tree, T is a level ≤ 2 tree. A *representation of* $Y \otimes T$ is a pair (R, ρ) such that

1. R is a level-3 tree;

2. ρ factors (R, Y, T);

3. $\operatorname{ran}(\rho) = \operatorname{desc}(Y, T, *)$;

Representations of $Y \otimes T$ are clearly mutually isomorphic.

As before, we shall regard

$$Y \otimes T$$

itself as a "level-3 tree" whose domain is the set of non-constant $(Y, T, *)$-descriptions and sends $\mathbf{B} \in \operatorname{desc}(Y, T, (Q, (d_i, q_i, P_i)_{1 \leq i \leq k}))$ to $(Q, (d_k, q_k, P_k))$. If Q is a level ≤ 2 tree, then $(Y \otimes T) \otimes Q$ is a "level-3 tree" whose domain consists of non-constant $(Y \otimes T, Q, *)$-descriptions. There is a natural isomorphism

$$\iota_{Y,T,Q}$$

between "level-3 trees" $(Y \otimes T) \otimes Q$ and $Y \otimes (T \otimes Q)$, defined as follows:

1. If $\mathbf{A} = ((\mathbf{B}, Z, \overrightarrow{(d,z,N)}), \psi) \in \operatorname{desc}(Y \otimes T, Q, U)$, $\mathbf{B} = (\mathbf{y}, \pi) \in \operatorname{desc}(Y, T, ($ $\mathbf{y} = (y, X, \overrightarrow{(e,x,W)})$, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}, \iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi)$.

2. If $\mathbf{A} = ((\mathbf{B}^\frown(-1, Z^+), Z^+, \overrightarrow{(d,z,N)}), \psi) \in \operatorname{desc}(Y \otimes T, Q, U)$, $\overrightarrow{(d,z,N)} = (d_i, z_i, N_i)_{1 \le i \le l}$, $\mathbf{B} = (\mathbf{y}, \pi) \in \operatorname{desc}(Y, T, (Z, \overrightarrow{(d,z,N)}))$, $\mathbf{y} = (y, X, (e_i, x_i,$ then

   (a) if y is of discontinuous type, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}^\frown(-1, X^+), \psi *_0 \pi)$, where $\psi *_0 \pi$ factors $(X^+, T \otimes Q, U)$, X^+ is a completion of $(X, (e_i, x_i))$, $\psi *_0 \pi$ extends $\iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi$, $\psi *_0 \pi(e_k, x_k) = \iota^{-1}_{T,Q,U}(2, \mathbf{t}_0, \tau)$, $\mathbf{t}_0 = ((-1), \{(0)\}, ((0)))$, $\tau((0)) = \psi(d_l, z_l)$;

   (b) if y is of continuous type, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}, \psi *_1 \pi)$, where $\psi *_1 \pi$ factors $(X, T \otimes Q, U)$, $\psi *_1 \pi$ extends $\iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi \restriction (\operatorname{dom}(X) \setminus \{(e_k, x_k)\})$, $\psi *_1 \pi(e_k, x_k) = \iota^{-1}_{T,Q,U}(2, \mathbf{t}^\frown(-1), \tau^+)$, where $\pi(e_k, x_k) = (2, \mathbf{t}, \tau)$, $\mathbf{t} = (t, S, (s_i)_{i \le m})$, τ^+ extends τ, $\tau^+(s_m) = \psi(d_l, z_l)$.

$\iota_{Y,T,Q}$ justifies the associativity of the $\otimes$ operator acting on level $(3, \le 2, \le 2)$ trees.

If $y \in \operatorname{dom}(Y)$, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)}) \in \operatorname{desc}(Y)$,

$$Y \otimes_y T$$

is the level-3 subtree of $Y \otimes T$ whose domain is $\operatorname{dom}(Y \otimes Q^0)$ plus all the $(Y, T, *)$-descriptions of the form $(\mathbf{y}, \tau)$. If π factors level ≤ 2 trees (T, Q), then

$$Y \otimes \pi$$

factors $(Y \otimes T, Y \otimes Q)$, where $Y \otimes \pi(\mathbf{y}, \psi) = (\mathbf{y}, (\pi \otimes U) \circ \psi)$ for $(\mathbf{y}, \psi) \in \operatorname{desc}(Y, T, U)$.

If ρ factors finite trees (R, Y, T), then ρ induces

$$\tilde{\rho}^T : \operatorname{desc}^{**}(R) \to \operatorname{desc}^{**}(Y)$$

as follows:

1. If $\mathbf{A} = (\emptyset, \emptyset, \emptyset)$, then $\tilde{\rho}^T(\mathbf{A}) = \mathbf{A}$.
2. If $\mathbf{A} = (\mathbf{r}, \psi, U)$, $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)})$ is of discontinuous type, $\rho(r) = (\mathbf{y}, \pi)$, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y}, (T \otimes \psi) \circ \pi)$.

3. If $\mathbf{A} = (\mathbf{r}, \psi, U)$, $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)})$ is of continuous type, $\overrightarrow{(d, q, P)} = (d_i, q_i, P_i)_{1 \leq i \leq l}$, $\rho(r^-) = (\mathbf{y}, \pi)$, $\mathbf{y} = (y, X, (e_i, x_i, W_i)_{1 \leq i \leq k})$,

   (a) if y is of discontinuous type, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y}^\frown(-1, X^+), \psi *_0 \pi)$, where $\psi *_0 \pi$ factors $(X^+, T \otimes U)$, $\psi *_0 \pi$ extends $(T \otimes \psi) \circ \pi$, $\psi *_0 \pi(e_k, x_k) = (2, \mathbf{t}_0, \tau)$, $\mathbf{t}_0 = ((-1), \{(0)\}, ((0)))$, $\tau((0)) = \psi(d_l, p_l)$;

   (b) if y is of continuous type, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y}, \psi *_1 \pi)$, where $\psi *_1 \pi$ factors $(X, T \otimes U)$, $\psi *_1 \pi$ extends $(T \otimes \psi) \circ (\pi \restriction \mathrm{dom}(X) \setminus \{(e_k, x_k)\})$, $\psi *_1 \pi(e_k, x_k) = (2, \mathbf{t}^\frown(-1), \tau^+)$, where $\pi(e_k, x_k) = (2, \mathbf{t}, \tau)$, $\mathbf{t} = (t, S, (s_i)_{i \leq m})$, τ^+ extends τ, $\tau^+(s_m) = \psi(d_l, p_l)$.

$\mathbf{A} \prec^R_* \mathbf{A}'$ iff $\tilde{\rho}^T(\mathbf{A}) \prec^Y_* \tilde{\rho}^T(\mathbf{A}')$; $\mathbf{A} \sim^R_* \mathbf{A}'$ iff $\tilde{\rho}^T(\mathbf{A}) \sim^Y_* \tilde{\rho}^T(\mathbf{A}')$. A purely combinatorial argument shows that if $R = Y \otimes T$, then for any $\mathbf{B} \in \mathrm{desc}^{**}(Y)$ there is $\mathbf{A} \in \mathrm{desc}^{**}(R)$ such that $\tilde{\rho}^T(\mathbf{A}) \sim^Y_* \mathbf{B}$.

Put $[\![\emptyset]\!]_R = \text{o.t.}(<^R)$. For $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)}) \in \mathrm{desc}^*(R)$, put

$$[\![\mathbf{r}]\!]_R = [\vec{\beta} \mapsto \|\vec{\beta} \oplus_R r\|_{<^R}]_{\mu^Q}.$$

If $\mathbf{r} \in \mathrm{desc}(R)$ is of discontinuous type, put $[\![r]\!]_R = [\![\mathbf{r}]\!]_R$. Note that if ρ factors

Π^1_3-wellfounded trees (R, Y), then $\llbracket r \rrbracket_R \leq \llbracket \rho(r) \rrbracket_Y$ for any $r \in \mathrm{dom}(R)$. We say that ρ *minimally factors* (R, Y) iff ρ factors (R, Y), R, Y are both Π^1_3-wellfounded and $\llbracket r \rrbracket_R = \llbracket \rho(r) \rrbracket_Y$ for any $r \in \mathrm{dom}(R)$. In particular, if Y is Π^1_3-wellfounded and T is Π^1_2-wellfounded, then $\mathrm{id}_{Y,*}$ minimally factors $(Y, Y \otimes T)$. In the assumption of Lemma **??**, if R, Y are Π^1_3-wellfounded and $\mathrm{ran}(\theta)$ is a $<^Y$-initial segment of $\mathrm{rep}(Y)$, its proof constructs ρ which minimally factors $(R, Y \otimes T)$.

A comparison theorem between Π^1_3-wellfounded trees.

Theorem 0.22. *Suppose R, Y are Π^1_3-wellfounded level-3 trees and $\llbracket \emptyset \rrbracket_R \leq \llbracket \emptyset \rrbracket_Y$. Then there exists (T, ρ) such that T is Π^1_2-wellfounded and ρ minimally factors $(R, Y \otimes T)$. Furthermore, if $\llbracket \emptyset \rrbracket_R < \llbracket \emptyset \rrbracket_Y$, we further obtain $\mathbf{B} \in \mathrm{dom}(Y \otimes T)$ such that $lh(\mathbf{B}) = 1$ and $\llbracket \emptyset \rrbracket_R = \llbracket \mathbf{B} \rrbracket_{Y \otimes T}$.*

$\mathcal{L} = \{\underline{\in}\}$ is the language of set theory. For a level-3 tree R, $\mathcal{L}^R$ is the expansion of $\mathcal{L}$ which consists of additional constant symbols $\underline{c_r}$ for each $r \in \mathrm{dom}(R)$.

For a level-3 tree R and a tuple of ordinals $\vec{\gamma} = (\gamma_r)_{r \in \mathrm{dom}(R)}$, the $\mathcal{L}$-structure

$M^-_{2,\infty}$ expands to the $\mathcal{L}^R$-structure

$$(M^-_{2,\infty}; \vec{\gamma})$$

whose constant $\underline{c_r}$ is interpreted as γ_r.

Assume $\mathbf{\Pi}^1_3$-determinacy. There is a club C in $\boldsymbol{\delta}^1_3$ such that

$$0^{3\#}$$

is a map sending a finite level-3 tree R to the complete consistent $\mathcal{L}^R$-theory $0^{3\#}(R)$, where $\ulcorner \varphi \urcorner \in 0^{3\#}(R)$ iff φ is an $\mathcal{L}^R$-formula and for all $\vec{\gamma} \in [C]^{R\uparrow}$,

$$(M^-_{2,\infty}; \vec{\gamma}) \models \varphi.$$

We introduce the following informal symbols arising from the proof of Lemma **?** that will occur in $\mathcal{L}$-formulas or $\mathcal{L}^R$-formulas for a level-3 tree R:

1. If Q is a finite level ≤ 2 tree, $\underline{j^Q}$ is the informal symbol so that $\underline{j^Q}(a) = b$ iff for any ξ cardinal cutpoint such that $\{a, b\} \in K|\xi$, the $Coll(\omega, \xi)$-generic extension satisfies $j^Q(\pi_{K|\xi,\infty}(a)) = \pi_{K|\xi,\infty}(b)$.
2. If π factors finite level ≤ 2 trees (X, T), $\underline{\pi^T}$ is the informal symbol so that $\underline{\pi^T}(a) = b$ iff for any ξ cardinal cutpoint such that $\{a, b\} \in K|\xi$, the

$Coll(\omega,\xi)$-generic extension satisfies $\pi^T(\pi_{K|\xi,\infty}(a)) = \pi_{K|\xi,\infty}(b)$.

3. If Q is a level ≤ 2 subtree of Q', Q' is finite, then $\underline{j^{Q,Q'}} = \underline{(\mathrm{id}_Q)^{Q,Q'}}$, where id_Q factors (Q,Q'), $\mathrm{id}_Q(d,q) = (d,q)$.

4. Corresponding to items 1-3, $\underline{j^Q_{\sup}}$, $\underline{\pi^T_{\sup}}$, $\underline{j^{Q,Q'}_{\sup}}$ stand for functions on ordinals that send α to $\sup(\underline{j^Q})''\alpha$, $\sup(\underline{\pi^T})''\alpha$, $\sup(\underline{j^{Q,Q'}})''\alpha$ respectively.

5. $\underline{S_3}$ is the informal symbol such that $(\emptyset,\emptyset) \in \underline{S_3}$ and $((R_i)_{i\leq n},(\alpha_i)_{i\leq n}) \in \underline{S_3}$ iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \mathrm{dom}(R_{i+1}) \setminus \mathrm{dom}(R_i)$, then $r_k = (r_l)^- \to \alpha_k < \underline{j^{(R_n)_{\mathrm{tree}}(r_k),(R_n)_{\mathrm{tree}}(r_l)}}(\alpha_l)$.

6. For $1 \leq n \leq \omega$, $\underline{u_n}$ is the symbol so that for any $\xi > \underline{u_n}$ cardinal cutpoint, the $Coll(\omega,\xi)$-generic extension satisfies $\pi_{K|\xi,\infty}(\underline{u_n}) = u_n$.

7. Suppose T is a finite level ≤ 2 tree. If $\mathbf{D} \in \mathrm{desc}(T,U)$, $\|\mathbf{D}\|_{\prec^{T,U}} = n$, then $\underline{\mathrm{seed}^{T,U}_{\mathbf{D}}} = \underline{u_{n+1}}$. If $(1,t) \in \mathrm{dom}(T)$, then $\underline{\mathrm{seed}^T_{(1,t)}} = \underline{\mathrm{seed}^{T,\emptyset}_{(1,t,\emptyset)}}$. If $(2,t) \in \mathrm{dom}(T)$, and ${}^2T[t] = (S,\vec{s})$, then $\underline{\mathrm{seed}^T_{(2,t)}} = \underline{\mathrm{seed}^{T,S}_{(2,(t,S,\vec{s}),\mathrm{id}_S)}}$. $\underline{\mathrm{seed}^T} = (\underline{\mathrm{seed}^T_{(d,t)}})_{(d,t)\in\mathrm{dom}(T)}$.

8. If k is a definable class function and W is a definable class, then $k(W) = \bigcup\{k(W \cap V_\alpha) : \alpha \in \mathrm{Ord}\}$.

9. If X, T, T' are finite level ≤ 2 trees, T is a subtree of T', $a \in \underline{j^X}(V)$, $d \in \{1, 2\}$, then

(a) $\underline{B^T_{X,a}} = \{\underline{\pi^{T \otimes Q}}(a) : Q$ finite level ≤ 2 tree, π factors $(X, T \otimes Q)\}$;

(b) $\underline{H^T_{X,a}}$ is the transitive collapse of the Skolem hull of $\underline{B^T_{X,a}} \cup \mathrm{ran}(\underline{j^T})$ in $\underline{j^T}(V)$ and $\underline{\phi^T_{X,a}} : \underline{H^T_{X,a}} \to \underline{j^T}(V)$ is the associated elementary embedding;

(c) $\underline{j^T_{X,a}} = (\underline{\phi^T_{X,a}})^{-1} \circ \underline{j^T}$;

(d) $\underline{j^{T,T'}_{X,a}} = (\underline{\phi^{T'}_{X,a}})^{-1} \circ \underline{j^{T,T'}} \circ \underline{\phi^T_{X,a}}$;

(e) $\underline{B^T_{1,a}} = \underline{B^T_{Q^0,a}} \cup \underline{B^T_{Q^1,a}}$, $\underline{B^T_{2,a}} = \underline{B^T_{Q^0,a}} \cup \underline{B^T_{Q^{20},a}} \cup \underline{B^T_{Q^{21},a}}$;

(f) $\underline{H^T_{d,a}}$ is the transitive collapse of the Skolem hull of $\underline{B^T_{d,a}} \cup \mathrm{ran}(\underline{j^T})$ in $\underline{j^T}(V)$ and $\underline{\phi^T_{d,a}} : \underline{H^T_{d,a}} \to \underline{j^T}(V)$ is the associated elementary embedding;

(g) $\underline{j^T_{d,a}} = (\underline{\phi^T_{d,a}})^{-1} \circ \underline{j^T}$;

(h) $\underline{j^{T,T'}_{d,a}} = (\underline{\phi^{T'}_{d,a}})^{-1} \circ \underline{j^{T,T'}} \circ \underline{\phi^T_{d,a}}$.

10. Suppose R is a level-3 tree.

(a) If $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)}) \in \operatorname{desc}^*(R)$, $\underline{c_{\mathbf{r}}}$ is the informal $\mathcal{L}^R$-symbol whose interpretation is

$$\underline{c_{\mathbf{r}}} = \begin{cases} \underline{j_{\sup}^{R_{\text{tree}}(r^-),Q}}(\underline{c_{r^-}}) & \text{if } \mathbf{r} \in \operatorname{desc}(R) \text{ of continuous type,} \\ \underline{c_r} & \text{if } \mathbf{r} \in \operatorname{desc}(R) \text{ of discontinuous type,} \\ \underline{j^{R_{\text{tree}}(r),Q}}(\underline{c_r}) & \text{if } \mathbf{r} \notin \operatorname{desc}(R). \end{cases}$$

(b) If T, U are finite level ≤ 2 trees and $\mathbf{B} = (\mathbf{r}, \pi) \in \operatorname{desc}(R, T, U)$, $\mathbf{r} \neq \emptyset$, then $\underline{c^T_{\mathbf{B}}}$ is the informal $\mathcal{L}^R$-symbol which stands for $\underline{\pi^{T,U}}(\underline{c_{\mathbf{r}}})$.

(c) If $\mathbf{A} = (\mathbf{r}, \pi, T) \in \operatorname{desc}^{**}(R)$, $\mathbf{r} \neq \emptyset$, then $\underline{c_{\mathbf{A}}}$ is the informal $\mathcal{L}^R$-symbol which stands for $\underline{\pi^T}(\underline{c_{\mathbf{r}}})$.

Suppose $\mathcal{M}, \mathcal{N}$ are countable Π^1_3-iterable mice. A map $\pi : \mathcal{M} \to \mathcal{N}$ is *essentially an iteration map* iff there are $\mathcal{P}$ and iteration maps $\psi_{\mathcal{M}} : \mathcal{M} \to$

$\mathcal{P}$, $\psi_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$ such that $\psi_{\mathcal{M}} = \psi_{\mathcal{N}} \circ \pi$. For $\alpha \in \mathcal{M}$, $\beta \in \mathcal{N}$, say that $(\mathcal{M}, \alpha) <_{DJ} (\mathcal{N}, \beta)$ iff either $\mathcal{M} <_{DJ} \mathcal{N}$ or there exist $\mathcal{P}$ and iteration maps $\psi_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$, $\psi_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$ such that $\psi_{\mathcal{M}}(\alpha) < \psi_{\mathcal{N}}(\beta)$.

Definition 0.23 (Level-3 EM blueprint)**.** A *pre-level-3 EM blueprint* is a function Γ sending any finite level-3 tree Y to a complete consistent $\mathcal{L}^Y$-theory $\Gamma(Y)$ which contains all of the following axioms:

1. ZFC + there is no inner model with two Woodin cardinals $+V = K+$ there is no strong cardinal $+V$ is closed under the $M_1^{\#}$-operator.
2. Suppose X, T, Q, Z are finite level ≤ 2 trees, π factors (X, T), ψ factors (T, Z).
   (a) $\underline{j^T} : V \to \underline{j^T}(V)$ is $\mathcal{L}$-elementary. $\underline{j^{Q^0}}$ is the identity map on V.
   (b) $\underline{\pi^T} : \underline{j^X}(V) \to \underline{j^T}(V)$ is $\mathcal{L}$-elementary. $\underline{j^{Q^0,T}} = \underline{j^T}$. $\underline{j^{T,T}}$ is the identity map on $\underline{j^T}(V)$.
   (c) $\underline{(\psi \circ \pi)^Z} = \underline{\psi^Z} \circ \underline{\pi^T}$.
   (d) $\underline{j^T} \circ \underline{j^Q} = \underline{j^{T \otimes Q}}$.

(e) $\underline{j^Q}(\underline{\pi^T}) = \underline{(Q \otimes \pi)^{Q\otimes T}}$.

(f) $\underline{\pi^T} \restriction \underline{j^{X\otimes Q}}(V) = \underline{(\pi \otimes Q)^{T\otimes Q}}$.

3. If ξ is a cardinal and strong cutpoint, then $V^{Coll(\omega,\xi)}$ satisfies the following: If U is a Π^1_2-wellfounded level ≤ 2 tree, then $K|\xi$ and $(\underline{j^U})^K(K|\xi)$ are countable Π^1_3-iterable mice and $(\underline{j^U})^K \restriction (K|\xi)$ is essentially an iteration map from $K|\xi$ to $(\underline{j^U})^K(K|\xi)$. Here $(\underline{j^U})^K$ stands for the direct limit map of $(\underline{j^{Z,Z'}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z'.

4. For any $y \in \mathrm{dom}(Y)$, "$\underline{c_y} \in \mathrm{Ord}$" is an axiom.

5. If $\mathbf{y} \prec^Y \mathbf{y}'$, then "$\underline{c_{\mathbf{y}}} < \underline{c_{\mathbf{y}'}}$" is an axiom; if $\mathbf{y} \sim^Y \mathbf{y}'$, then "$\underline{c_{\mathbf{y}}} = \underline{c_{\mathbf{y}'}}$" is an axiom.

A level-3 EM blueprint is a pre-level-3 EM blueprint satisfying the *coherency property*: if R, Y, T are finite, ρ factors (R, Y, T), then for each $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$, for each $r_1, \ldots, r_n \in \mathrm{dom}(R)$,

$$\ulcorner \varphi(\underline{c_{r_1}}, \ldots, \underline{c_{r_n}}) \urcorner \in \Gamma(R)$$

iff

$$\ulcorner \underline{j^T}(V) \models \varphi(\underline{c^T_{\rho(r_1)}}, \ldots, \underline{c^T_{\rho(r_n)}})\urcorner \in \Gamma(Y).$$

In particular, if Γ is a level-3 EM blueprint, ρ_0 factors (R, Y), then $\mathrm{id}_{Y,*} \circ \rho_0$ factors (R, Y, Q^0), so by coherency, $\ulcorner\varphi(\underline{c_{r_1}}, \ldots)\urcorner \in \Gamma(R)$ iff $\ulcorner\varphi(\underline{c_{\rho_0(r_1)}}, \ldots)\urcorner \in \Gamma(Y)$. This degenerates to the usual indiscernability of the (level-1) EM blueprint.

Lemma 0.24. *Assume* $\mathbf{\Pi}^1_3$*-determinacy. Then* $0^{3\#}$ *is a level-3 EM blueprin*

We say that the *upward closure* of $A \subseteq (\omega^{<\omega})^{<\omega}$ is

$$\{r \in (\omega^{<\omega})^{<\omega} : \exists a \in A (r \subseteq a)\}.$$

The upward closure does not apply to subcoordinates of $a \in A$. For instance, $b \subsetneq a(\mathrm{lh}(a) - 1)$ does not imply that $a^{-\frown}(b)$ is in the upward closure of A. For a level-3 tree R and nodes $s_1, \ldots, s_n, s'_1, \ldots, s'_n$ in $\mathrm{dom}(R)$,

$$\vec{s'} \text{ is an } R\text{-}shift \text{ of } \vec{s}$$

iff there are a level-3 tree S and maps ρ, ρ' factoring (S, R) such that $\mathrm{ran}(\rho)$ is the upward closure of $\vec{s}$, $\mathrm{ran}(\rho')$ is the upward closure of $\vec{s'}$, and $\rho^{-1}(s_i) =$

$(\rho')^{-1}(s'_i)$ for any i.

As with the usual treatment of $0^\#$, a level-3 EM blueprint Γ admits an $\mathcal{L}$-Skolemized conservative extension. That means, since ZFC $+V = K$ is a part of the axioms, so is "there is a $\Sigma_1^{\mathcal{L}}$-definable wellordering of the universe". Thus, to each $\mathcal{L}$-formula $\varphi(v, w_1, \ldots, w_n)$ we may attach a definable $\mathcal{L}$-Skolem term $\tau_\varphi(w_1, \ldots, w_n)$ so that the formula $\forall w_1 \ldots w_n\ (\exists v\, \varphi(v, w_1, \ldots, w$ $\varphi(\tau_\varphi(w_1, \ldots, w_n), w_1, \ldots, w_n))$ belongs to $\Gamma(R)$, for any R.

If Y is an infinite level-3 tree, put

$$\Gamma(Y) = \bigcup\{\Gamma(R) : R \text{ is a finite level-3 subtree of } Y\}.$$

By coherency, $\Gamma(R) \subseteq \Gamma(R')$ whenever $R \subseteq R'$ are finite. Hence by compactness, $\Gamma(Y)$ is a complete consistent $\mathcal{L}^Y$-theory. The usual argument of EM models with order indiscernibles carries over to obtain a unique up to isomorphism $\mathcal{L}^Y$-structure

$$\mathcal{M}_{\Gamma,Y} = (M; \underline{\in}^M, \underline{c_t}^M : t \in \mathrm{dom}(Y)).$$

such that $\mathcal{M}_{\Gamma,Y}$ is $\mathcal{L}$-Skolem generated by $\{\underline{c_t}^M : t \in \mathrm{dom}(Y)\}$, and

$$\mathcal{M}_{\Gamma,Y} \models \Gamma(Y).$$

$\mathcal{M}_{\Gamma,Y}$ is called the *EM model* associated to Γ and Y. When $\underline{\in}^{\mathcal{M}_{\Gamma,Y}}$ is wellfounded, $\mathcal{M}_{\Gamma,Y}$ is identified with its transitive collapse. Since $\mathcal{M}_{\Gamma,Y}$ is a model of $V = K$, the extender sequence on $K^{\mathcal{M}_{\Gamma,Y}}$ is definable over $\mathcal{M}_{\Gamma,Y}$, this allows us to sometimes treat $\mathcal{M}_{\Gamma,Y}$ as a structure in the language of premice.

If $\mathcal{L}^*$ is a first-order language expanding $\mathcal{L}$, $\mathcal{N}$ is an $\mathcal{L}^*$-structure satisfying axioms 1-3 in Definition 0.23, we make the following notations:

1. If T is a finite level ≤ 2 tree, then $j^T_{\mathcal{N}} = (\underline{j^T})^{\mathcal{N}}$, $\mathcal{N}^T = (\underline{j^T}(V))^{\mathcal{N}}$ is an $\mathcal{L}^*$-structure so that $j^T_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T$ is $\mathcal{L}^*$-elementary.
2. If π factors finite level ≤ 2 trees (X, T), then $\pi^T_{\mathcal{N}} = (\underline{\pi^T})^{\mathcal{N}}$. If T, T' are finite level ≤ 2 trees, T is a subtree of T', then $j^{T,T'}_{\mathcal{N}} = (\underline{j^{T,T'}})^{\mathcal{N}}$.
3. If T is a level ≤ 2 tree, then $\mathcal{N}^T$ is the direct limit of $(\mathcal{N}^{T'}, j^{T',T''}_{\mathcal{N}} : T', T''$ finite subtrees of T, T' a finite subtree of $T'')$ and $j^T_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T$ is the

direct limit map; if T' is a finite subtree of T, then $j^{T',T}_{\mathcal{N}} : \mathcal{N}^{T'} \to \mathcal{N}^T$ is the tail of the direct limit map. The wellfounded part of $\mathcal{N}^T$ is always assumed to be transitive.

4. If π factors level ≤ 2 trees (X, T), then $\pi^T_{\mathcal{N}} : \mathcal{N}^X \to \mathcal{N}^T$ is the factor map between direct limits.

5. If X is a finite level ≤ 2 tree, $a \in \mathcal{N}^X$, $d \in \{1, 2\}$
   (a) if T, T' are finite level ≤ 2 trees, T is a subtree of T', then $\mathcal{N}^T_{X,a} = (\underline{H^T_{X,a}})^{\mathcal{N}}$, $j^T_{X,a,\mathcal{N}} = (\underline{j^T_{X,a}})^{\mathcal{N}}$, $\phi^T_{X,a,\mathcal{N}} = (\underline{\phi^T_{X,a}})^{\mathcal{N}}$, $j^{T,T'}_{X,a,\mathcal{N}} = (\underline{j^{T,T'}_{X,a}})^{\mathcal{N}}$, $\mathcal{N}^T_{d,a} = (\underline{H^T_{d,a}})^{\mathcal{N}}$, $j^T_{d,a,\mathcal{N}} = (\underline{j^T_{d,a}})^{\mathcal{N}}$, $\phi^T_{d,a,\mathcal{N}} = (\underline{\phi^T_{d,a}})^{\mathcal{N}}$, $j^{T,T'}_{d,a,\mathcal{N}} = (\underline{j^{T,T'}_{d,a}})^{\mathcal{N}}$;
   (b) if T is a level ≤ 2 tree, then $\mathcal{N}^T_{X,a}$ is the natural direct limit, $j^T_{X,a,\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T_{X,a}$ is the direct limit map, $\phi^T_{X,a,\mathcal{N}} : \mathcal{N}^T_{X,a} \to \mathcal{N}^T$ is the natural factoring map between direct limits; if T' is a finite subtree of T, then $j^{T',T}_{X,a,\mathcal{N}} : \mathcal{N}^{T'}_{X,a} \to \mathcal{N}^T_{X,a}$ is the tail of the direct limit map; similarly

define $\mathcal{N}^T_{d,a}$, $j^T_{d,a,\mathcal{N}}$, $\phi^T_{d,a,\mathcal{N}}$, $j^{T',T}_{d,a,\mathcal{N}}$.

If Γ is a level-3 EM blueprint and R is Π^1_3-wellfounded, we make further notations:

1. If T is a level ≤ 2 tree, then $\mathcal{M}^T_{\Gamma,Y} = (\mathcal{M}_{\Gamma,Y})^T$, $j^T_{\Gamma,Y} = j^T_{\mathcal{M}_{\Gamma,Y}}$.
2. If T is a finite subtree of T', then $j^{T,T'}_{\Gamma,Y} = j^{T,T'}_{\mathcal{M}_{\Gamma,Y}}$.
3. If π factors (X,T), then $\pi^T_{\Gamma,Y} = \pi^T_{\mathcal{M}_{\Gamma,Y}}$.
4. If T is a finite subtree of T', $y \in \mathrm{dom}(Y)$, $X = Y_{\mathrm{tree}}(y)$, $d \in \{1,2\}$, then $c_{\Gamma,Y,y} = (\underline{c_y})^{\mathcal{M}_{\Gamma,Y}}$, $\mathcal{M}^T_{\Gamma,Y,y} = (\mathcal{M}_{\Gamma,Y})^T_{X,c_{\Gamma,Y,y}}$, $j^T_{\Gamma,Y,y} = j^T_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $\phi^T_{\Gamma,Y,y} = \phi^T_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $j^{T,T'}_{\Gamma,Y,y} = j^{T,T'}_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $\mathcal{M}^T_{\Gamma,R^d,*} = (\mathcal{M}_{\Gamma,R^d})^T_{d,c_{\Gamma,R^d,((0}}$ $j^T_{\Gamma,R^d,*} = j^T_{d,c_{\Gamma,R^d,((0))},\mathcal{M}_{\Gamma,R^d}}$, $\phi^T_{\Gamma,R^d,*} = \phi^T_{d,c_{\Gamma,R^d,((0))},\mathcal{M}_{\Gamma,R^d}}$, $j^{T,T'}_{\Gamma,R^d,*} = j^{T,T'}_{d,c_{\Gamma,R^d,((0))}}$
5. If $\mathbf{B} \in \mathrm{desc}(Y,T',*)$ and T' is a finite subtree of T, then $c^T_{\Gamma,Y,\mathbf{B}} = j^{T',T}_{\Gamma,Y}(\underline{c^{T'}_{\mathbf{B}}})^{\mathcal{M}_{\Gamma,Y}}$.

By coherency, if ρ factors (R, Y, T), then ρ induces an elementary embedding

$$\rho_\Gamma^{Y,T} : \mathcal{M}_{\Gamma,R} \to \mathcal{M}_{\Gamma,Y}^T$$

where

$$\rho_\Gamma^{Y,T}(\tau^{\mathcal{M}_{\Gamma,R}}(c_{\Gamma,R,r_1}, \ldots)) = \tau^{\mathcal{M}_{\Gamma,Y}^T}(c_{\Gamma,Y,\rho(r_1)}^T, \ldots).$$

If ρ factors (R, Y), then ρ induces

$$\rho_\Gamma^Y : \mathcal{M}_{\Gamma,R} \to \mathcal{M}_{\Gamma,Y}$$

where $\rho_\Gamma^Y(\tau^{\mathcal{M}_{\Gamma,R}}(c_{\Gamma,R,r_1}, \ldots)) = \tau^{\mathcal{M}_{\Gamma,Y}}(c_{\Gamma,Y,\rho(r_1)}, \ldots)$.

Recall that wellfoundedness of a (level-1) EM blueprint is a Π^1_2 condition, stating that for every countable ordinal α, the EM model generated by order indiscernibles of order type α is wellfounded. Its higher level analog is called iterability, which is a Π^1_4 condition.

Definition 0.25. Let Γ be a level-3 EM blueprint. Γ is *iterable* iff for any Π^1_3-wellfounded level-3 tree Y, $\mathcal{M}_{\Gamma,Y}$ is a Π^1_3-iterable mouse.

We start to introduce the remarkability property of a level-3 EM blueprint.

For $r, s \in \omega^{<\omega}$, define $r <_0 s$ iff $r(0) <_{BK} s(0)$, $r \leq_0^R s$ iff $r(0) \leq_{BK} s(0)$.

If $\vec{r} = (r_i)_{1\le i\le n}$ is a tuple of nodes in $\omega^{<\omega}$, define $\vec{r} <_0 s$ iff $r_i <_0 s$ for any i. Similarly define $\vec{r} \le_0 s$, $\vec{r} <_0 \vec{s}$, etc.

Definition 0.26 (Unboundedness)**.** A level-3 EM blueprint Γ is *unbounded* iff for any level-3 tree R, if τ is an $\mathcal{L}$-Skolem term, $\{t, r_1, \ldots, r_m\} \subseteq \mathrm{dom}(R)$, $\vec{r} <_0 t$, then $\Gamma(R)$ contains the formula

$$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}) \in \mathrm{Ord} \to \tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}) < \underline{c_t}.$$

Definition 0.27 (Weak remarkability)**.** A level-3 EM blueprint Γ is *weakly remarkable* iff Γ is unbounded and for any level-3 tree R, if τ is an $\mathcal{L}$-Skolem term, $\vec{r} \cup \vec{s} \cup \vec{s'} \cup \{t\} \subseteq \mathrm{dom}(R)$, $\vec{r} <_0 t \le_0 \vec{s} \le_0 \vec{s'}$, $\vec{s'}$ is an R-shift of $\vec{s}$, $\mathrm{lh}(t) = 1$, then $\Gamma(R)$ contains the formula

$$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s_1}}, \ldots, \underline{c_{s_n}}) < \underline{c_t} \to \\ \tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s_1}}, \ldots, \underline{c_{s_n}}) = \tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s'_1}}, \ldots, \underline{c_{s'_n}}).$$

If R is a level-3 tree, $t \in \mathrm{dom}(R)$, $\mathrm{lh}(t) = 1$, let

$$R \restriction t = R \restriction \{r \in \mathrm{dom}(R) : r <_0 t\}.$$

A level-3 tree R is said to be *universal above t* iff $t \in \mathrm{dom}(R)$, $\mathrm{lh}(t) =$

1, and for any level-3 tree S, if $S \restriction t'$ is isomorphic to $R \restriction t$ via π and $\mathrm{dom}(S) \setminus \mathrm{dom}(S \restriction t')$ is finite, then there is a map ρ factoring (S, R) that extends π. Clearly, for any R, there is (R', t) such that $R' \restriction t$ is isomorphic to R and R' is universal above t. If R is Π^1_3-wellfounded, we may further demand that R' is Π^1_3-wellfounded.

A level-3 tree R is *universal based on* Y iff there is $t \in \mathrm{dom}(R)$ such that $\mathrm{lh}(t) = 1$, R is universal above t and $R \restriction t$ is isomorphic to Y. Suppose Γ is a weakly remarkable level-3 EM blueprint. For a level-3 tree Y, if R is universal based on Y, $t \in \mathrm{dom}(R)$, $\mathrm{lh}(t) = 1$, $R \restriction t$ is isomorphic to Y, put

$$\mathcal{M}^*_{\Gamma,Y} = (K|\underline{c_t})^{\mathcal{M}_{\Gamma,R}}.$$

$\mathcal{M}^*_{\Gamma,Y}$ is well-defined up to an isomorphism. Its wellfounded part is transitivized. There are cofinally many cardinal strong cutpoints in $\mathcal{M}^*_{\Gamma,Y}$. Similarly, for a level ≤ 2 tree T, define

$$\mathcal{M}^{*,T}_{\Gamma,Y} = (K|\underline{c_t})^{\mathcal{M}^T_{\Gamma,R}}.$$

Hence, $\mathcal{M}^{*,T}_{\Gamma,Y} = (\mathcal{M}^*_{\Gamma,Y})^T$. If ρ factors (Y, Y'), R' is universal above R, then

$\rho_\Gamma^{*,Y'} = \rho_\Gamma^{R'} \restriction \mathcal{M}^*_{\Gamma,Y}$. If ρ factors (Y, Y', T), R' is universal above R, then $\rho_\Gamma^{*,Y',T} = \rho_\Gamma^{Y',T} \restriction \mathcal{M}^*_{\Gamma,Y}$.

A Π^1_3-iterable mouse $\mathcal{P}$ is *full* iff for any strong cutpoint η of $\mathcal{P}$, for any Π^1_3-iterable mouse $\mathcal{Q}$ extending $\mathcal{P}|\eta$ which is sound and projects to η, $\mathcal{Q} \trianglelefteq \mathcal{P}$.

An ordinal $\alpha < \omega_1$ is *ω_1-represented by T* iff $(1,(0)) \in \operatorname{dom}(T)$ and $[\![1,(0)]\!]_T = \alpha$. $\alpha < u_2$ is *u_2-represented by T* iff $(2,((0))) \in \operatorname{dom}(T)$ and $[\![2,((0))]\!]_T = \alpha$.

Definition 0.28 (Remarkability)**.** A weakly remarkable level-3 EM blueprint Γ is *remarkable* iff

1. $\Gamma(R^0)$ contains the axiom "$\underline{c_{((0))}}$ is not measurable".
2. $\Gamma(R^1)$ contains the following axiom: if ξ is a cardinal and strong cutpoint, $c = \underline{c_{((0))}}$, $b = (\underline{\phi^{Q^1}_{1,c}})^{-1}(c)$, then $V^{Coll(\omega,\xi)}$ satisfies the following:
   (a) If α is ω_1-represented by both T and T', then $((\underline{j^T_{1,c}})^K(K|\xi), ((\underline{j^{Q^1,T}_{1,c}})^K(b))$ $((\underline{j^{T'}_{1,c}})^K(K|\xi), ((\underline{j^{Q^1,T'}_{1,c}})^K(b))$. Here $(\underline{j^U_{1,c}})^K$ stands for the direct limit of

$(\underline{j^{Z,Z'}_{1,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\underline{j^{Q^1,U}_{1,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^{Q^1}_{1,c}})^K(K)$ to $(\underline{j^{U}_{1,c}})^K(K)$.

(b) Let $F(\alpha) = \pi_{(\underline{j^T_{1,c}})^K(K|\xi),\infty}((\underline{j^{Q^1,T}_{1,c}})^K(b))$ for α represented by T. Then $\sup_{\alpha<\omega_1} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

3. $\Gamma(R^2)$ contains the following axiom: if ξ is a cardinal and strong cutpoint, $e \in \{0,1\}$, $c = \underline{c_{((0))}}$, $b = (\underline{\phi^{Q^{2e}}_{1,c}})^{-1}(c)$, then $V^{Coll(\omega,\xi)}$ satisfies the following:

(a) If α is u_2-represented by both T and T', then $((\underline{j^T_{2,c}})^K(K|\xi),((\underline{j^{Q^{2e},T}_{2,c}})^K(b))$ $((\underline{j^{T'}_{2,c}})^K(K|\xi),((\underline{j^{Q^{2e},T'}_{2,c}})^K(b))$. Here $(\underline{j^U_{2,c}})^K$ stands for the direct limit of $(\underline{j^{Z,Z'}_{2,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\underline{j^{Q^{2e},U}_{2,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^{Q^{2e}}_{2,c}})^K(K)$ to $(\underline{j^{U}_{2,c}})^K(K)$.

(b) Let $F(\alpha) = \pi_{(\underline{j^T_{2,c}})^K(K|\xi),\infty}((\underline{j^{Q^{2e},T}_{2,c}})^K(b))$ for α represented by T. Then $\sup_{\alpha<u_2} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

Definition 0.29 (Level ≤ 2 correctness)**.** A level-3 EM blueprint Γ is *level* ≤ 2 *correct* iff for each finite level-3 tree Y, for each $y \in \text{dom}(Y)$, putting $X = Y_{\text{tree}}(y)$, $\Gamma(Y)$ contains the following axiom:

If $c = \underline{c_y}$, $b = (\underline{\phi^X_{X,c}})^{-1}(c)$, $\xi > c$ is a cardinal and strong cutpoint, then $V^{Coll(\omega,\xi)}$ satisfies the following:

1. If $\vec{\alpha} = ({}^d\alpha_x)_{(d,x)\in\text{dom}(X)}$ is represented by both T and T', then $((\underline{j^T_{X,c}})^K(K|\xi), (\underline{j^{X,T}_{X,c}})^K(b)) \sim_{DJ} ((\underline{j^{T'}_{X,c}})^K(K|\xi), (\underline{j^{X,T'}_{X,c}})^K(b))$. Here $(\underline{j^U_{X,c}})$ stands for the direct limit of $(\overline{j^{Z,Z'}_{X,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\overline{j^{X,U}_{X,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^X_{X,c}})^K(K)$ to $(\underline{j^U_{X,c}})^K(K)$.
2. Let $F(\vec{\alpha}) = \pi_{(\underline{j^T_{X,c}})^K(K|\xi),\infty}((\underline{j^{X,T}_{X,c}})^K(b))$ for $\vec{\alpha}$ represented by T. Then

$$[F]_{\mu^X} = \pi_{K|\xi,\infty}(c).$$

Theorem 0.30. *Assume* $\mathbf{\Pi}^1_3$*-determinacy. Then* $0^{3\#}$ *is the unique iterable, remarkable, level* ≤ 2 *correct level-3 EM blueprint.*

Thank you for your attention!