On Stein's Method for Multivariate Self-Decomposable Laws

Benjamin Arras (Laboratoire Paul Painlevé, Université de Lille)

Joint Works With Christian Houdré (GaTech, Atlanta)

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Stein's Method (Gaussian Case $\approx$ 1970)

- Three Main Steps:
  - Functional Characterization: depends on the limiting law
  - Differential/Difference/Integro-Differential Equation
  - Discerning: depends on the sequential structure

First Step: Let $X = Z \sim \mathcal{N}(0,1)$, then for all $f \in \mathcal{F}$,

$$\mathbb{E}(Xf(X)) = \mathbb{E}(f'(X)),$$

CONVERSELY

If this last identity holds for all $f \in \mathcal{F}$, then it characterizes normal laws among all zero mean unit variance ones, i.e., $X \sim \mathcal{N}(0,1)$.

(First Step is sometimes called Stein's Lemma: For the direct implication it is an integration by parts $\int xf(x)e^{-x^2/2}dx = \int f'(x)e^{-x^2/2}dx$; for the converse "need to know the solution".)

Second Step: Given Y, if

$$\mathbb{E}(Yf(Y)) \approx \mathbb{E}(f'(Y)),$$

then $Y \overset{Law}{\approx} Z$.
Stein's equation, for all $x \in \mathbb{R}$ and any test function h,

$$f_h'(x) - xf_h(x) = h(x) - \mathbb{E}h(Z),$$

with $\|f_h^{(k)}\|_\infty \leq C_k$ ($k = 0, 1$) so that

$$\mathbb{E}h(Z) \approx \mathbb{E}h(Y) \quad \text{if} \quad \mathbb{E}(f_h'(Y) - Yf_h(Y)) \approx 0.$$

Applications

A Simple Example

Let $(X_i)_{i\geq 1}$ be iid and such that $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = 1$ and $\mathbb{E}(|X_1|^3) < +\infty$. For any $n \geq 1$, let $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$. Then,

$$W_1(S_n, Z) := \sup_{h \in Lip(1)} |\mathbb{E}h(S_n) - \mathbb{E}h(Z)| \leq \frac{C\,\mathbb{E}|X_1|^3}{\sqrt{n}},$$

for some $C > 0$.

$$d_K(S_n, Z) := \sup_{h = \mathbf{1}_{(-\infty, x]}} |\mathbb{E}h(S_n) - \mathbb{E}h(Z)| \leq \frac{C\,\mathbb{E}|X_1|^3}{\sqrt{n}},$$

for some $C > 0$.

Extensions/Interactions

- More complex dependency structures where characteristic functions methods have shortcomings.
- Several variation on the method: multivariate normal approximation, Poisson approximation, Compound Poisson approximation, geometric approximation, exponential approximation, various Stein-type identities, many ad-hoc methods, functional characterization changes from one law to another.
- Statistical Physics, Spin Glasses, Spanning Trees, Concentration Inequalities, Random Matrices, Number Theory.
- Malliavin Calculus, Functional Inequalities, Optimal Transport, Dirichlet forms

The Poisson Case (L. Chen $\approx$ 1975)

Let $X \sim \mathcal{P}(\lambda)$, then for any f on $\mathbb{N}$,

$$\mathbb{E}Xf(X) = \lambda \mathbb{E}f(X+1), \textit{i.e.},$$

$$\mathbb{E}Xf(X) = \mathbb{E}X\,\mathbb{E}f(X) + \mathbb{E}X\mathbb{E}(f(X+1) - f(X)).$$

Conversely, the above identity characterizes the Poisson law.
This then leads to a difference equation (to be solved) and Poisson Convergence Theorems with rates follow.

Common Framework for Gaussian and Poisson Laws?

Infinitely divisible laws!

Infinitely Divisible (ID) Laws on $\mathbb{R}^d$

Definition

Let $X \sim \mu$ have characteristic function φ. Then, X is infinitely divisible (ID) if, for each $n \geq 1$, there exists a characteristic function φ_n such that, for all $\xi \in \mathbb{R}^d$,

$$\varphi(\xi) = (\varphi_n(\xi))^n .$$

Equivalently,

$$X \stackrel{\mathcal{L}}{=} X_{1,n} + ... + X_{n,n},$$

where $(X_{i,n})_{1 \leq i \leq n}$ are iid with characteristic function φ_n.

Examples

- Discrete Laws : Poisson, Geometric, Negative Binomial, etc.
- Absolutely Continuous Laws : Gaussian, Stable, Laplace, Gamma, second Wiener chaos, double Pareto, Gumbel, cube of normal, etc.
- Laws with bounded support not ID

Some Very Classical Results

Closure Under Weak Convergence

Let $(X_n)_{n\geq 1}$ be a sequence of ID r.v.s converging in law to X_∞. Then, X_∞ is ID.

Approximation by sequence of compound Poisson laws

Let X be *ID* with c.f. φ. For each $n \geq 1$, let X_n be a r.v. defined via the c.f. φ_n given by

$$\varphi_n(\xi) = \exp\left(n\left(\varphi(\xi)^{\frac{1}{n}} - 1\right)\right), \quad \xi \in \mathbb{R}^d.$$

Then, $X_n \underset{n\to+\infty}{\overset{\mathrm{Law}}{\longrightarrow}} X$.

A Classical Limit Theorem

Let $(r_n)_{n\geq 1}$ be a sequence of integers $r_n \underset{n\to+\infty}{\longrightarrow} +\infty$

$\forall \varepsilon > 0,\ \max_{1\leq k\leq r_n} \mathbb{P}(\|Z_{nk}\| > \varepsilon) \underset{n\to+\infty}{\longrightarrow} 0$

$\forall n \geq 1,\ (Z_{n1}, Z_{n2}, ..., Z_{nr_n})$ independent

Z_{11}	Z_{12}	...	Z_{1r_1}
Z_{21}	Z_{22}	...	Z_{2r_2}
...	...	...	...
Z_{n1}	Z_{n2}	...	Z_{nr_n}
...	...	...	...

Kolmogorov-Khintchine ~ 1937

If there exist a sequence of vectors of $\mathbb{R}^d$ $(c_n)_{n\geq 1}$ and a probability measure μ such that

$$\sum_{k=1}^{r_n} Z_{nk} + c_n \underset{n\to+\infty}{\overset{\text{Law}}{\longrightarrow}} \mu$$

Then, μ is ID.

Self-Decomposable Laws on $\mathbb{R}^d$

Definition

A r.v. $X \sim \mu$ with c.f. φ is self-decomposable (SD) if for any $0 < c < 1$ there exists a c.f. φ_c s.t.

$$\varphi(\xi) = \varphi(c\xi)\varphi_c(\xi), \quad \xi \in \mathbb{R}^d \qquad , \textit{i.e.},$$

$$X \overset{Law}{=} cX + X_c,$$

with X_c independent of X, i.e.,

$$\mu = T_c(\mu) * \mu_c,$$

where $T_c(\mu)(B) = \mu(B/c)$, B Borel set of $\mathbb{R}^d$.

Remarks

- SD laws are ID
- SD laws are absolutely continuous with respect to Lebesgue measure
- Some classical examples: stable as well as gamma laws but many others!

Another Classical Limit Theorem

- $(b_n)_{n\geq 1}$ is a sequence of strictly positive reals
- $(Z_k)_{k\geq 1}$ independent s.t., $\forall \varepsilon > 0$, $\max_{1\leq k\leq n} \mathbb{P}(b_n \|Z_k\| > \varepsilon) \underset{n\to+\infty}{\longrightarrow} 0$

Khintchine-Lévy ~ 1938

If there exist a sequence of reals $(c_n)_{n\geq 1}$ and a probability measure μ such that

$$S_n = b_n \sum_{k=1}^{n} Z_k + c_n \underset{n\to+\infty}{\overset{\text{Law}}{\longrightarrow}} \mu$$

then μ is self-decomposable (subclass of ID laws) (when μ is non-degenerate, then necessarily $b_{n+1}/b_n \to 1$ and $b_n \underset{n\to+\infty}{\to} 0$. Moreover, the converse is also true, i.e., any self decomposable μ is the weak limit of sums as above.

Stein's Method For ID (SD) Laws

C. Stein, 1986

"Two other cases that seem likely to introduce interesting new features but no insuperable difficulties are infinitely divisible laws and the multivariate normal case."

Questions // Program

- ID versions of characterizing identity?
- Quantitative versions of the previous results?
- Which quantities allow for a control of usual distances implying weak convergence?
- Explicit rates of convergence?

Representation

Lévy-Khintchine Representation

Let X be a r.v. with law μ and characteristic function φ. Then X is ID if and only if for all $\xi \in \mathbb{R}^d$

$$\varphi(\xi) = \exp\left(i\langle b;\xi\rangle - \frac{1}{2}\langle \xi;\Sigma\xi\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u;\xi\rangle} - 1 - i\langle u;\xi\rangle\, \mathbf{1}_{\|u\|\leq 1} \right) \nu(du) \right),$$

$b \in \mathbb{R}^d$, $\Sigma \geq 0$, and ν is a positive Borel measure on $\mathbb{R}^d$ s.t. $\int_{\mathbb{R}^d} (1 \wedge \|u\|^2)\nu(du) < +\infty$ and $\nu(\{0\}) = 0$. $X \sim ID(b, \Sigma, \nu)$.

Remark

- No Gaussian component $\iff \Sigma = 0$
- For $d = 1$, Poisson $\iff ID(\lambda, 0, \lambda\delta_1)$
- $\int_{\|u\|\geq 1} \|u\|\nu(du) < +\infty$, then,
  $\varphi(\xi) = \exp\left(i\langle \mathbb{E}X;\xi\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u;\xi\rangle} - 1 - i\langle u;\xi\rangle \right) \nu(du) \right)$
- $\int_{\|u\|\leq 1} \|u\|\nu(du) < +\infty$, then, $\varphi(\xi) = \exp\left(i\langle b_0;\xi\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u;\xi\rangle} - 1 \right) \nu(du) \right)$, with $b_0 = b - \int_{\|u\|\leq 1} u\nu(du)$

Characterizing ID Laws With $\mathbb{E}\|X\| < +\infty$

Theorem [B. A. and C.Houdré, 2017]

Let $d = 1$. Let X with $\mathbb{E}|X| < +\infty$ and ν be a Lévy measure on $\mathbb{R}$ such that $\int_{|u|\geq 1} |u|\nu(du) < +\infty$. Then,

$$\mathrm{Cov}(X, f(X)) = \mathbb{E}\int_{-\infty}^{+\infty} (f(X+u) - f(X))\, u\nu(du),$$

for all f, bounded Lipschitz on $\mathbb{R}$, **if and only if** X is an ID random vector with Lévy measure ν (and $b = \mathbb{E}X - \int_{|u|>1} u\nu(du)$).

Theorem [B. A. and C.Houdré, 2019+]

Let X with $\mathbb{E}\|X\| < +\infty$ and ν be a Lévy measure on $\mathbb{R}^d$ such that $\int_{\|u\|\geq 1} \|u\|\nu(du) < +\infty$. Then,

$$\mathbb{E}Xf(X) = \mathbb{E}X\,\mathbb{E}f(X) + \mathbb{E}\int_{\mathbb{R}^d} (f(X+u) - f(X))\, u\nu(du),$$

for all f, bounded Lipschitz on $\mathbb{R}^d$, **if and only if** X is an ID random vector with Lévy measure ν (and $b = \mathbb{E}X - \int_{\|u\|>1} u\nu(du)$).

Some Observations

Corollary [B. A. and C.Houdré, 2019+]

Let $\alpha \in (1, 2)$ and let ν_α be the Lévy measure

$$\nu_\alpha(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)\frac{c_{\alpha,d}}{r^{\alpha+1}}dr\sigma(dx),$$

with $c_{\alpha,d} > 0$ and with σ the uniform measure on $\mathbb{S}^{d-1}$. Let $X_\alpha \sim ID(b_\alpha, 0, \nu_\alpha)$ with $b_\alpha = -\int_{\|u\|\geq 1} u\nu_\alpha(du)$. Then, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathbb{E}X_\alpha f(X_\alpha) = \mathbb{E}\int_{\mathbb{R}^d} (f(X_\alpha + u) - f(X_\alpha))\, u\nu_\alpha(du) \underset{\alpha \to 2^-}{\longrightarrow} \mathbb{E}Zf(Z) = \mathbb{E}\nabla(f)(Z),$$

where $Z \sim \mathcal{N}(0, I_d)$.

Some Observations

Theorem [S. Cohen and J. Rosinski, 2007, *Bernoulli*]

Let $\varepsilon > 0$ and $X_\varepsilon \sim ID(b_\varepsilon, 0, \nu_\varepsilon)$ with $b_\varepsilon = -\int_{\|u\|\geq 1} u\nu_\varepsilon(du)$ such that

$$\Sigma_\varepsilon = \int_{\mathbb{R}^d} uu^t \nu_\varepsilon(du),$$

is non-singular. Then, as $\varepsilon \to 0^+$, $\tilde{X}_\varepsilon = \Sigma_\varepsilon^{-1/2} X_\varepsilon \overset{Law}{\to} Z \sim \mathcal{N}(0, I_d) \iff$

$$\int_{\langle \Sigma_\varepsilon^{-1} u; u\rangle > \kappa} \langle \Sigma_\varepsilon^{-1} u; u\rangle \nu_\varepsilon(du) \underset{\varepsilon \to 0^+}{\longrightarrow} 0, \quad \kappa > 0.$$

Corollary [B. A. and C.Houdré, 2019+]

Let the above convergence hold. Then, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathbb{E}\tilde{X}_\varepsilon f(\tilde{X}_\varepsilon) = \mathbb{E}\int_{\mathbb{R}^d} (f(\tilde{X}_\varepsilon + u) - f(\tilde{X}_\varepsilon)) u\tilde{\nu}_\varepsilon(du) \underset{\varepsilon \to 0^+}{\longrightarrow} \mathbb{E}Zf(Z) = \mathbb{E}\nabla(f)(Z),$$

with $Z \sim \mathcal{N}(0, I_d)$.

Characterizing SD Laws With $\int_{\|u\| \le 1} \|u\| \nu(du) < +\infty$

Theorem [B. A. and C.Houdré, 2019+]

Let X be a random vector in $\mathbb{R}^d$. Let $b \in \mathbb{R}^d$, let ν be a Lévy measure with $\int_{\|u\| \le 1} \|u\| \nu(du) < +\infty$, and with

$$\nu(du) = \mathbb{I}_{(0,+\infty)}(r) \mathbb{I}_{\mathbb{S}^{d-1}}(x) \frac{k(r)}{r} dr \sigma(dx),$$

where σ is a finite positive measure on $\mathbb{S}^{d-1}$ and where $k(r)$ is a nonnegative continuous function decreasing in $r \in (0, +\infty)$ and such that

$$\lim_{\varepsilon \to 0^+} \varepsilon k(\varepsilon) = 0, \quad \lim_{R \to +\infty} k(R) = 0.$$

Let $\tilde{\nu}$ be defined by $\tilde{\nu}(du) = \mathbb{I}_{(0,+\infty)}(r) \mathbb{I}_{\mathbb{S}^{d-1}}(x)(-dk(r))\sigma(dx)$. Then,

$$\mathbb{E}\langle X; \nabla(f)(X) \rangle = \mathbb{E}\langle b_0; \nabla(f)(X) \rangle + \int_{\mathbb{R}^d} (f(X+u) - f(X)) \, \tilde{\nu}(du),$$

where $b_0 = b - \int_{\|u\| \le 1} u \nu(du)$, for all $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ **if and only if** X is SD with b and Lévy measure ν.

Characterizing SD Laws

Theorem [B. A. and C.Houdré, 2019+]

Let X be a random vector in $\mathbb{R}^d$. Let $b \in \mathbb{R}^d$, let ν be a Lévy measure with

$$\nu(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)\frac{k(r)}{r}dr\sigma(dx),$$

where σ is a finite positive measure on $\mathbb{S}^{d-1}$ and where $k(r)$ is a nonnegative continuous function decreasing in $r \in (0,+\infty)$ and such that

$$\lim_{\varepsilon\to 0^+} \varepsilon k(\varepsilon) = k(1), \quad \lim_{R\to+\infty} k(R) = 0.$$

Let $\tilde{\nu}$ be defined by $\tilde{\nu}(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)(-dk(r))\sigma(dx)$. Then,

$$\mathbb{E}\langle X; \nabla(f)(X)\rangle = \mathbb{E}\langle \tilde{b}; \nabla(f)(X)\rangle + \mathbb{E}\int_{\mathbb{R}^d} \left(f(X+u) - f(X) - \langle \nabla(f)(X); u\rangle \mathbb{1}_{\|u\|\leq 1}\right)\tilde{\nu}(du),$$

with $\tilde{b} = b - k(1)\int_{\mathbb{S}^{d-1}} x\sigma(dx)$, for all $f \in \mathcal{S}\left(\mathbb{R}^d\right)$, **if and only if** X is SD with b and Lévy measure ν.

Characterizing SD Laws

Remarks

- Extend to the general case $k_x(r)$ (under additional technical conditions)
- In the stable case, $\tilde{\nu}(du) = \alpha\nu(du)$
- Direct part of the proof: a truncation procedure and an integration by parts (boundary terms!)
- Converse part of the proof: PDE techniques in the Fourier domain
- Let X be SD with b, ν such that $\int_{\|u\|\geq 1} \|u\|\nu(du) < +\infty$ and with k as in the previous theorem. For all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathbb{E}\langle X; \nabla(f)(X)\rangle = \mathbb{E}\langle \tilde{b}; \nabla(f)(X)\rangle + \mathbb{E}\int_{\mathbb{R}^d} \left(f(X+u) - f(X) - \langle \nabla(f)(X); u\rangle \mathbb{1}_{\|u\|\leq 1}\right)\tilde{\nu}(du)$$

An integration by parts

$$\mathbb{E}\langle X; \nabla(f)(X)\rangle = \mathbb{E}\langle \mathbb{E}X; \nabla(f)(X)\rangle + \mathbb{E}\int_{\mathbb{R}^d} \langle \nabla(f)(X+u) - \nabla(f)(X); u\rangle \nu(du).$$

Stein's Equation For SD Laws with finite first moment

Let $X \sim ID(b, 0, \nu)$ be SD such that $\mathbb{E}\|X\| < +\infty$, with c.f. φ and with ν such that

$$\sup_{x \in S^{d-1}} k_x(a^+) < +\infty, \quad a > 0.$$

Let $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $M_0(h) = \sup_{x \in \mathbb{R}^d} |h(x)| \leq 1$, $M_1(h) = \sup_{x \in \mathbb{R}^d} \|\nabla(h)(x)\|_{op} \leq 1$, $M_2(h) = \sup_{x \in \mathbb{R}^d} \|D^2(h)(x)\|_{op} \leq 1$.

$$\langle (\mathbb{E}X - x); \nabla(f_h)(x) \rangle + \int_{\mathbb{R}^d} \langle \nabla(f_h)(x+u) - \nabla(f_h)(x); u \rangle \nu(du) = h(x) - \mathbb{E}h(X), \quad x \in \mathbb{R}^d.$$

Semigroup methods to solve the equation:

$$f_h(x) = -\int_0^{+\infty} (P_t(h)(x) - \mathbb{E}h(X))dt, \quad x \in \mathbb{R}^d$$

with

$$P_t(h)(x) = \int_{\mathbb{R}^d} h(xe^{-t} + y) d\mu_t(y)$$

where μ_t has characteristic function $\varphi_t(\xi) = \varphi(\xi)/\varphi(e^{-t}\xi)$. Moreover,

$$M_1(f_h) \leq 1, \qquad M_2(f_h) \leq \frac{1}{2}.$$

Stein's Equation For SD Laws without finite first moment

Let $X \sim ID(b, 0, \nu)$ be SD with c.f. φ such that ν

$$\nu(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)\frac{k_x(r)}{r}dr\sigma(dx),$$

with $k_x(r)$ continuous in $r \in (0, +\infty)$, continuous in $x \in \mathbb{S}^{d-1}$, with

$$\lim_{r \to 0^+} r^2 k_x(r) = 0, \quad \lim_{r \to +\infty} k_x(r) = 0, \quad x \in \mathbb{S}^{d-1},$$

Assume that

- There exists $\varepsilon \in (0,1)$ such that $\mathbb{E}\|X\|^\varepsilon < +\infty$.
- There exists $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 \in (0,1)$ such that

$$\gamma_1 = \sup_{t \geq 0}\left(e^{\beta_1 t}\int_{(1,+\infty)\times\mathbb{S}^{d-1}}\frac{k_x(e^t r)}{r}dr\sigma(dx)\right) < +\infty,$$

$$\gamma_2 = \sup_{t \geq 0}\left(e^{\beta_2 t}\int_{(0,1)\times\mathbb{S}^{d-1}} r k_x(e^t r)dr\sigma(dx)\right) < +\infty,$$

and that,

$$\gamma_3 = \sup_{t \geq 0}\left(e^{-(1-\beta_3)t}\left\|\int_{\mathbb{S}^{d-1}} x\left(\int_1^{e^t} k_x(r)dr\right)\sigma(dx)\right\|\right) < +\infty.$$

Stein's Equation For SD Laws without finite first moment

Assume also that, X_t, $t \geq 0$, with c. f. $\varphi_t(\xi) = \varphi(\xi)/\varphi(e^{-t}\xi)$ satisfies:

$$\sup_{t \geq 0} \mathbb{E} \|X_t\|^{\varepsilon} < +\infty.$$

Then, for all $h \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ with $M_0(h) \leq 1$, $M_1(h) \leq 1$ and $M_2(h) \leq 1$,

$$f_h(x) = -\int_0^{+\infty} (P_t(h)(x) - \mathbb{E}h(X))dt, \quad x \in \mathbb{R}^d$$

exists with $M_1(f_h) \leq 1$, $M_2(f_h) \leq 1/2$ and is a strong solution to

$$\langle \tilde{b} - x; \nabla(f_h)(x)\rangle + \int_{\mathbb{R}^d} \Big(f_h(x+u) - f_h(x) - \langle \nabla(f_h)(x); u\rangle \mathbb{1}_{\|u\| \leq 1} \Big) \tilde{\nu}(du) = h(x) - \mathbb{E}h(X),$$

where $\tilde{\nu}$ is given

$$\tilde{\nu}(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)(-dk_x(r))\sigma(dx),$$

and where $\tilde{b} = b - \int_{\mathbb{S}^{d-1}} k_x(1) x \sigma(dx)$.

Stein's Equation For SD Laws

Remarks

- Stable laws with $\alpha \in (0,1]$
- If the above assumptions hold and X such that

$$\int_{\|u\|\leq 1} \|u\|\nu(du) < +\infty, \quad \lim_{\varepsilon\to 0^+} \varepsilon k_x(\varepsilon) = 0, \quad x \in \mathbb{S}^{d-1},$$

then, f_h is a strong solution to

$$\langle b_0 - x; \nabla(f_h)(x)\rangle + \int_{\mathbb{R}^d} \Big(f_h(x+u) - f_h(x)\Big)\tilde{\nu}(du) = h(x) - \mathbb{E}h(X).$$

- If the above assumptions hold and X such that

$$\int_{\|u\|\geq 1} \|u\|\nu(du) < +\infty, \qquad \lim_{R\to +\infty} R k_x(R) = 0, \quad \lim_{\varepsilon\to 0^+} \varepsilon^2 k_x(\varepsilon) = 0, \quad x \in \mathbb{S}^{d-1},$$

then, f_h is a strong solution to

$$\langle \mathbb{E}X - x; \nabla(f_h)(x)\rangle + \int_{\mathbb{R}^d} \langle \nabla(f_h)(x+u) - \nabla(f_h)(x); u\rangle \nu(du) = h(x) - \mathbb{E}h(X).$$

Why Does This Work?

Decomposability property

- $\mu \in M_1(\mathbb{R}^d)$

$$D(\mu) = \{c \in (0,1), \mu = T_c(\mu) * \mu_c,\ \mu_c \in M_1(\mathbb{R}^d)\}$$

- If μ is SD then $D(\mu) = (0,1)$. In particular, $(e^{-t})_{t>0} \subset D(\mu)$.
- Setting $\mu_t := \mu_{e^{-t}}$, then

$$\mu_{t+s} = \mu_t * T_{e^{-t}}(\mu_s) \quad s, t > 0.$$

- Then, $P_t(h)(x) = \int_{\mathbb{R}^d} h(xe^{-t} + y) d\mu_t(y)$ defines a semigroup on $\mathcal{C}_b(\mathbb{R}^d)$ whose invariante measure is μ.
- Strong links between Urbanik decomposability semigroup (Urbanik, Jurek, Bunge) and Generalized Mehler semigroup (Röckner, Bogachev, Schmuland, Lescot, Wang.).

An Application To Extreme Value Theory

Let X be a Gumbel random variable with distribution function $F(x) := \exp\left(-\exp(-x)\right)$, for $x \in \mathbb{R}$, and Lévy Khintchine representation

$$\varphi(\xi) = \exp\left(i\xi\gamma + \int_0^{+\infty}\left(e^{i\xi u} - 1 - i\xi u\right)\frac{e^{-u}}{u(1-e^{-u})}du\right), \quad \xi \in \mathbb{R},$$

where γ is the Euler constant. Let $(Y_k)_{k\geq 1}$ be a i.i.d. sequence such that $Y_1 \sim \mathrm{Exp}(1)$.

Fact 1: $\max\left(Y_1, \ldots, Y_n\right) - \log n \xrightarrow[n\to+\infty]{Law} X$.

Fact 2: $S_n := \max\left(Y_1, \ldots, Y_n\right) \overset{Law}{=} \sum_{j=1}^{n} \frac{Y_j}{j}$.

Theorem, [B. A. and C.Houdré, 2019]

Then, for all $n \geq 1$

$$d_{W_2}(S_n, X) := \sup_{\substack{h\in\mathcal{C}_c^\infty(\mathbb{R}), M_0(h)\leq 1 \\ M_1(h), M_2(h)\leq 1}} |\mathbb{E}h(S_n) - \mathbb{E}h(X)| \leq \frac{C}{n},$$

for some $C > 0$ independent of n.

Sketch of Proof: 1

Let $h \in \mathcal{C}_c^\infty(\mathbb{R})$ with $M_0(h), M_1(h), M_2(h) \leq 1$.

$$\begin{aligned}|\mathbb{E}h(S_n) - \mathbb{E}h(X)| &= \left|\mathbb{E}(\gamma - S_n){f_h}'(S_n) + \int_0^{+\infty} \left({f_h}'(S_n + u) - {f_h}'(S_n)\right) \frac{e^{-u}}{(1-e^{-u})} du\right| \\ &\leq \left|\mathbb{E}(\mathbb{E}S_n - S_n){f_h}'(S_n) + \int_0^{+\infty} \left({f_h}'(S_n + u) - {f_h}'(S_n)\right) \frac{e^{-u}}{(1-e^{-u})} du\right| \\ &\quad + \left|\gamma - \mathbb{E}S_n\right|.\end{aligned}$$

First, for all $n \geq 1$

$$\left|\gamma - \mathbb{E}S_n\right| = \left|\gamma + \ln n - \sum_{k=1}^{n} \frac{1}{k}\right| \leq \frac{C_1}{n},$$

for some $C_1 > 0$.

Sketch of Proof: 2

Second, for all $n \geq 1$

$$\mathbb{E}S_n {f_h}'(S_n) = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E} Y_k {f_h}'(S_{n,k} + k^{-1} Y_k) - \ln n \, \mathbb{E} {f_h}'(S_n),$$

$$= \sum_{k=1}^{n} \frac{1}{k} \int_0^{+\infty} e^{-u} \mathbb{E} {f_h}'(S_n + k^{-1} u) du - \ln n \, \mathbb{E} {f_h}'(S_n).$$

Thus,

$$\left| \mathbb{E}(\mathbb{E}S_n - S_n) {f_h}'(S_n) + \int_0^{+\infty} \left({f_h}'(S_n + u) - {f_h}'(S_n) \right) \frac{e^{-u}}{(1 - e^{-u})} du \right|$$

$$= \left| \mathbb{E} \int_0^{+\infty} \left({f_h}'(S_n + u) - {f_h}'(S_n) \right) \frac{e^{-nu}}{e^u - 1} du \right| \leq \frac{1}{2} \int_0^{+\infty} \frac{e^{-nu}}{e^u - 1} u du.$$

Sketch of Proof: 3

Observe that, for all $n \geq 1$

$$\int_0^{+\infty} \frac{e^{-nu}}{e^u - 1} u du = \zeta(2, n+1),$$

where ζ is the Hurwitz zeta function given by

$$\zeta(2, n+1) = \sum_{k=0}^{+\infty} \frac{1}{(k+n+1)^2},$$

and, as $n \to +\infty$

$$\zeta(2, n+1) \approx \frac{C_2}{n}. \quad \square$$

An Application To Poincaré-type Inequalities for SD laws

Definition, [B. A. and C.Houdré, 2019]

Let ν be a Lévy measure on $\mathbb{R}^d$. Let Y be a centered random vector with law μ_Y. A Stein kernel of Y with respect to ν is a measurable function τ_Y from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that,

$$\int_{\mathbb{R}^d} \langle y; f(y) \rangle \mu_Y(dy) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \langle f(y+u) - f(y); \tau_Y(y+u) - \tau_Y(y) \rangle \nu(du) \right) \mu_Y(dy),$$

for all $\mathbb{R}^d$-valued test function f for which both sides of the previous equality are well defined. The Stein's discrepancy of μ_Y with respect to $\mu_X \sim ID(-\int_{\|u\| \geq 1} u\nu(du), 0, \nu)$ is given by

$$S(\mu_Y || \mu_X) = \inf \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\tau_Y(y+u) - \tau_Y(y) - u\|^2 \nu(du) \mu_Y(dy) \right)^{1/2},$$

where the infimum is taken over all Stein kernels of Y with respect to X, and is equal to $+\infty$ if no such Stein kernel exists.

An Application To Poincaré-type Inequalities for SD laws

Let $X \sim \mu_X$ be SD centered with finite second moment and with ν such that

$$\nu(du) = \mathbb{1}_{(0,+\infty)}(r)\mathbb{1}_{\mathbb{S}^{d-1}}(x)\frac{k_x(r)}{r}dr\sigma(dx),$$

with

$$\sup_{x \in S^{d-1}} k_x(a^+) < +\infty, \quad a > 0.$$

Let $Y \sim \mu_Y$ be centered such that

- $\nu * \mu_Y << \mu_Y$
- $\mathbb{E}\|Y\|^2 = \int_{\mathbb{R}^d} \|u\|^2 \nu(du) < +\infty$
- There exists $U_Y \geq 1$, for all regular $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$$\mathbb{E}\|f(Y) - \mathbb{E}f(Y)\|^2 \leq U_Y \mathbb{E}\int_{\mathbb{R}^d} \|f(Y+u) - f(Y)\|^2 \nu(du).$$

Theorem, [B. A. and C.Houdré, 2019]

Then,

$$d_{W_2}(\mu_X, \mu_Y) \leq \frac{1}{2}\left(\int_{\mathbb{R}^d} \|u\|^2 \nu(du)\right)\sqrt{U_Y - 1}.$$

Sketch of Proof: 1

Let $h \in \mathcal{H}_2 \cap \mathcal{C}_c^\infty(\mathbb{R}^d)$.

$$-\langle x; \nabla(f_h)(x)\rangle + \int_{\mathbb{R}^d} \langle \nabla(f_h)(x+u) - \nabla(f_h)(x); u\rangle \nu(du) = h(x) - \mathbb{E}h(X), \qquad x \in \mathbb{R}^d,$$

and thus,

$$\mathbb{E}\left(-\langle Y; \nabla(f_h)(Y)\rangle + \int_{\mathbb{R}^d} \langle \nabla(f_h)(Y+u) - \nabla(f_h)(Y); u\rangle \nu(du)\right) = \mathbb{E}h(Y) - \mathbb{E}h(X).$$

Now,

$$\mathbb{E}h(Y) - \mathbb{E}h(X) = \mathbb{E}\left(\int_{\mathbb{R}^d} \langle \nabla(f_h)(Y+u) - \nabla(f_h)(Y); u - \tau_Y(Y+u) + \tau_Y(Y)\rangle \nu(du)\right).$$

Sketch of Proof: 2

By Cauchy-Schwarz,

$$|\mathbb{E}h(Y) - \mathbb{E}h(X)| \leq \frac{1}{2}\sqrt{\int_{\mathbb{R}^d} \|u\|^2 \nu(du)}\sqrt{\mathbb{E}\int_{\mathbb{R}^d} \|\tau_Y(Y+u) - \tau_Y(Y) - u\|^2 \nu(du)}.$$

Next expanding the square,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d} \|\tau_Y(Y+u) - \tau_Y(Y) - u\|^2 \nu(du) &= \mathbb{E}\int_{\mathbb{R}^d} \|\tau_Y(Y+u) - \tau_Y(Y)\|^2 \nu(du) \\ &\quad - \int_{\mathbb{R}^d} \|u\|^2 \nu(du),\end{aligned}$$

and using the energy estimate

$$\mathbb{E}\int_{\mathbb{R}^d} \|\tau_Y(Y+u) - \tau_Y(Y)\|^2 \nu(du) \leq U_Y \int_{\mathbb{R}^d} \|u\|^2 \nu(du). \quad \square$$

Final Results: Remarks and Perspectives

- General ID case (with or without finite first moment) ?
- Quantitative results for m-dependent sequences converging to ID laws ?
- Links with functional inequalities and spectral problems for non-local Dirichlet forms ? (work in progress)
- Infinite Dimension ? Interactions with the geometry of Banach Spaces

Thank Your Attention !