Fluctuations in a general preferential attachment model

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1920 – 2016

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→ Barabási and Albert (1999) proposed a different dynamic model

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satisfying a power law for $f(k) = \gamma k + \beta$, $\gamma, \beta \in [0,1)$
or a stretched exponential for $f(k) = \gamma k^{\alpha}$, $\alpha \in (0,1), \gamma > 0$

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$$\mathbb{P}(n+1 \text{ connects to } j \,|\, \mathcal{G}_n) = \frac{f(\deg_n^-(j))}{n},$$

where $\deg_n^-(j)$ is the indegree of vertex j at time n

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where $\deg_n^-(j)$ is the indegree of vertex j at time n and $f : \mathbb{N}_0 \to (0, \infty)$ is monotonically increasing such that $f(n) \leq n+1$.

Preferential Attachment Model: Examples

- fixed outdegree – exactly one edge each step:
  $d_0 = 1$, add one vertex and set $f(k) = \frac{k+1+\delta}{n(2+\delta)}$ for a fixed parameter $\delta > -1$.
  close to Bollobás, Riordan, Spencer and Tusnády (2001); Hofstad (2017)

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- spatial attachment:
  on a hypercube; add an edge with probability p if it lies in the *sphere of influence* of an older vertex which itself depends on the degree
  Aiello, Bonato, Cooper, Janssen and Prałat (2008)

Example

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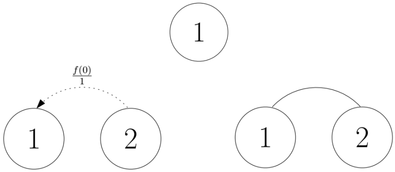

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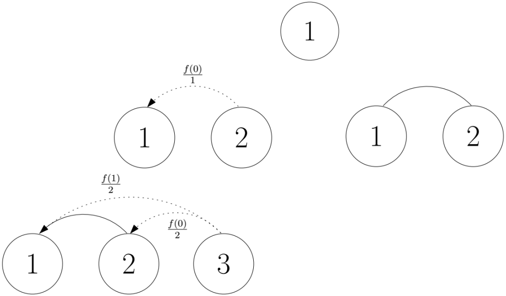

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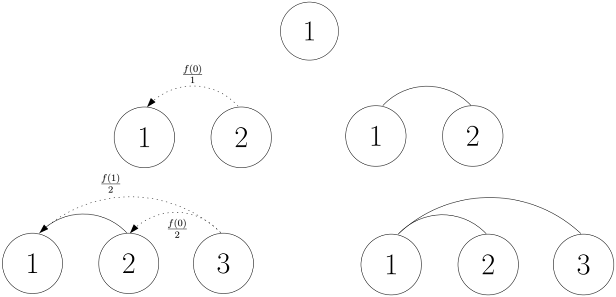

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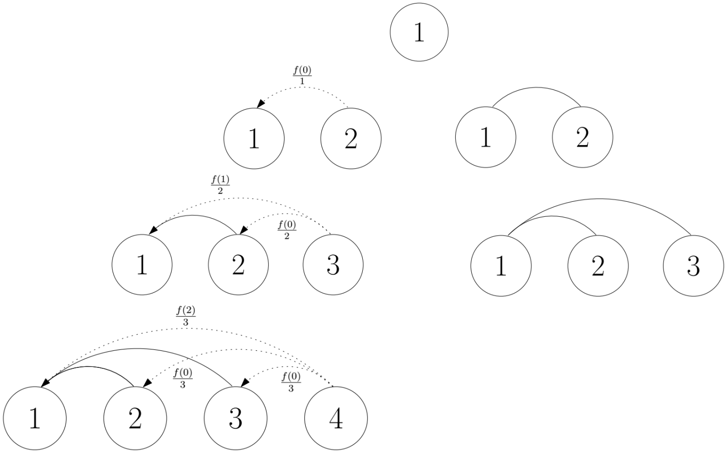
1
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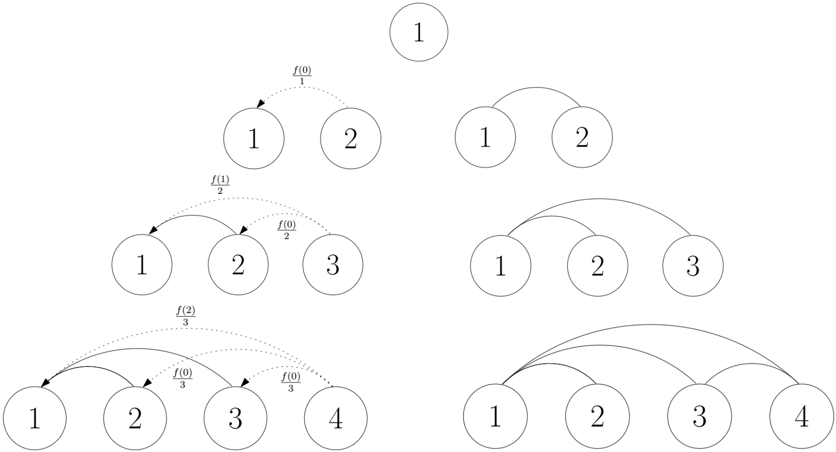

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Let W_n denote the indegree of a uniformly chosen vertex at time n in a preferential attachment model satisfying our assumptions.
Suppose further that there exists $k_* \in \mathbb{N}_0$ such that $f(k) > k$ for all $k < k_*$ and $f(k) \leq k$ for all $k \geq k_*$.

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satisfied e.g. for all sublinear models such that $\max_k \big(f(k+1) - f(k)\big) < 1$, see Dereich, Mörters (2013)

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Then, there exists a constant $C > 0$ such that for all $n \geq 2$

$$d_{\mathrm{TV}}(W_n, W) \leq C \, \frac{\log(n)}{n},$$

where $W \sim \mu = (\mu_k)_k = \left(\frac{1}{1+f(k)} \prod_{i=0}^{k-1} \frac{f(i)}{1+f(i)}\right)_k$.

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e.g. $f(k) = k + \gamma$: the distribution has power law exponent 2 and no finite mean: weaker convergence

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$$\mathbb{P}(W_n = k) = \mathbb{P}\big(deg^-(U_n) = k\big) \xrightarrow{n\to\infty} \mu_k = \frac{1}{1+f(k)} \prod_{i=0}^{k-1} \frac{f(i)}{1+f(i)}$$

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$$\mathbb{P}(N_Y = k) = \prod_{i=0}^{k-1} \frac{f(i)}{1+f(i)} - \prod_{i=0}^{k} \frac{f(i)}{1+f(i)} = \mu_k$$

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- for all $g : \mathbb{N}_0 \to \mathbb{R}$

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- Stein operator: Choose

$$\begin{aligned}\mathcal{A}g(k) :=& f(k)\big(g(k+1) - g(k)\big) + g(0) - g(k) \\ =& f(k)\Delta g(k) + g(0) - g(k)\end{aligned}$$

for $\Delta g(k) = g(k+1) - g(k)$.

Stein Equation for PA

With

$$\mathcal{A}g(k) = f(k)\Delta g(k) + g(0) - g(k)$$

and

$$\mu = (\mu_k) = \Big(\frac{1}{1+f(k)} \prod_{i=0}^{k-1} \frac{f(i)}{1+f(i)} \Big)_k$$

we have

$$W \sim \mu \quad \Leftrightarrow \quad \mathbb{E}[\mathcal{A}g(W)] = 0$$

for all $g : \mathbb{N}_0 \to \mathbb{R}$ such that $\mathbb{E}[g(W)] < \infty$.

Solution to Stein's Equation for PA

- Let (Z_t) denote a Markov chain with generator $\mathcal{A}$ solution of $\mathcal{A}g_A = h - \mu(h)$ for any $h = \mathbb{1}_A$, with $A \subset \mathbb{N}_0$ is given by

$$g_A(k) = -\int_0^\infty \left(\mathbb{E}_k h(Z_t) - \int h d\mu\right) dt$$

if it exists.
Barbour (1988), Götze (1991)

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- here: The solution can be checked using Kolmogorov's backward equation for Markov chains and a small coupling argument
- smoothness estimate:

$$f(k)\Delta g(k) \leq 1 \quad \text{for all } k \in \mathbb{N}_0$$

  following techniques in Brown, Xia (2001)

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- dynamic way of generating a uniform random variable on $\{1, \ldots, n\}$ cf. Ford (2009)

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- For any n, $J_n \sim U(\{1, \dots, n\})$
- $$W_n := \deg^-(U_n) \stackrel{d}{=} X_n := \deg^-(J_n)$$

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- For any n, $J_n \sim U(\{1, \ldots, n\})$
- $$W_n := \deg^-(U_n) \stackrel{d}{=} X_n := \deg^-(J_n)$$
- (X_n) is a time-inhomogeneous, discrete Markov chain, with generator

$$\frac{f(k)}{n+1}\Delta g(k) + \frac{1}{n+1}(g(0) - g(k))$$

Bounding the Stein Operator

- $W_n \stackrel{d}{=} X_n := \deg^-(J_n)$
- transition probabilities given for any $i \geq 1$ as

$$\mathbb{P}(X_{n+1} = j \,|\, X_n = i) = \begin{cases} \frac{f(i)}{n+1} & \text{if } j = i+1, \\ & \\ & \end{cases}$$

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Bounding the Stein Operator

- aim: Bound

$$
\begin{aligned}
&|\mathbb{E}[\mathcal{A}g_A(W_{n+1})]| = |\mathbb{E}\mathcal{A}g_A(X_{n+1})]| \\
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- Let $h : \mathbb{N}_0 \to \mathbb{R}$ be such that $h(0) = 0$, then

$$\mathbb{E}\left[h(X_{n+1})\right] = \mathbb{E}\left[\mathbb{E}\left[h(X_{n+1}) \mid X_n\right]\right]$$

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- iteration and $X_1 = 0$ yields

$$
\mathbb{E}\left[h(X_{n+1})\right] = \frac{1}{n+1}\sum_{\ell=1}^{n}\mathbb{E}[f(X_\ell)\Delta h(X_\ell)].
$$

Bounding the Stein solution

- $\mathbb{E}\left[h(X_{n+1})\right] = \dfrac{1}{n+1}\sum_{\ell=1}^{n}\mathbb{E}[f(X_\ell)\Delta h(X_\ell)]$
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$$
\begin{aligned}
\mathbb{E}[\mathcal{A}g_A(X_{n+1})] &= \mathbb{E}[h(X_{n+1})] + v_A(0) \\
&= \frac{1}{n+1}\sum_{\ell=1}^{n}\Big(\mathbb{E}[f(X_\ell)\Delta v_A(X_\ell)] - \mathbb{E}[f(X_\ell)\Delta g_A(X_\ell)]\Big) + v_A(0) \\
&= \frac{1}{n+1}\sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1} f(k)\Delta v_A(k)\mathbb{P}(X_\ell = k) \\
&\quad - \frac{1}{n+1}\sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1}(v_A(k) - v_A(0))\mathbb{P}(X_\ell = k) + \frac{v_A(0)}{n+1}.
\end{aligned}
$$

Bounding the Stein solution

- For the second sum, we write

$$\sum_{k=0}^{\ell-1}(v_A(k) - v_A(0))\mathbb{P}(X_\ell = k) = \sum_{k=0}^{\ell-1}\sum_{i=0}^{k-1}\Delta v_A(i)\mathbb{P}(X_\ell = k)$$
$$= \sum_{i=0}^{\ell-1}\Delta v_A(i)\sum_{k=i+1}^{\ell-1}\mathbb{P}(X_\ell = k) = \sum_{i=0}^{\ell-1}\Delta v_A(i)\mathbb{P}(X_\ell \geq i+1).$$

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- Therefore,

$$\mathbb{E}[\mathcal{A}g_A(X_{n+1})]=\frac{1}{n+1}\sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1}f(k)\Delta v_A(k)\mathbb{P}(X_\ell=k)$$
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- Thus, we used the Markov structure to show

$$\mathbb{E}\left[\mathcal{A}g_A(X_{n+1})\right] = \frac{1}{n+1}\Big(\sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1}\Delta v_A(k)h(k,\ell) + v_A(0)\Big),$$

where

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$$v_A(k) := f(k)\Delta g_A(k) \quad \text{smoothness estimate!}$$
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Bounding the Stein solution

- By discrete integration by parts formula and $h(\ell, \ell) = 0$:

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-

$$\begin{aligned}
&\left|\mathbb{E}[\mathcal{A}g_A(X_{n+1})]\right| \\
&\leq \frac{|v_A(0)|}{n+1} + \left|\frac{1}{n+1}\sum_{\ell=1}^{n}\left(\left(\sum_{k=0}^{\ell-1} v_A(k+1)\Delta_k h(k,\ell)\right) + v_A(0)h(0,\ell)\right)\right| \\
&\leq \frac{|v_A(0)|}{n+1} + \frac{2}{n+1}\sup_k |v_A(k)| \sum_{\ell=1}^{n} \sup_{k\leq \ell-1} h(k,\ell).
\end{aligned}$$

Bounding the Stein solution

And finally

$$
\begin{aligned}
d_{\mathrm{TV}}(W_n, W) &= \sup_{A \subset \mathbb{N}_0} \left| \mathbb{E}[\mathcal{A} g_A(X_{n+1})] \right| \\
&\leq \frac{1}{n+1} + \frac{2}{n+1} \sum_{\ell=1}^{n} \sup_{k \leq \ell - 1} h(k, \ell) \\
&\leq \begin{cases} C \, \frac{\log(n)}{n} & \text{for slowly growing } f \\ C \, \frac{1}{n^{1-\gamma}} & \text{for } f(k) = k + \gamma. \end{cases}
\end{aligned}
$$

□

Theorem 3 (Betken, D., Ortgiese, 2019)

Let D_n denote the outdegree of vertex n in a preferential attachment model satisfying our assumptions.

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- Suppose $f(k) \leq \gamma k + 1$ for $\gamma \in (0, 1)$. Then there exists $C > 0$ such that

$$d_{\mathrm{TV}}(D_n, Po(\lambda_n)) \leq C \begin{cases} \frac{1}{n+1}, & \text{for } 0 < \gamma < \frac{1}{2}, \\ \frac{\log(n)}{n}, & \text{for } \gamma = \frac{1}{2}, \\ n^{-2(1-\gamma)}, & \text{for } \frac{1}{2} < \gamma < 1, \end{cases}$$

where $\lambda_n = \mathbb{E}\left[f(W_{n-1})\right]$.

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- If $f(k) = \gamma k + \beta$ for $\gamma \in (0,1), \beta \in [0,1]$, then

$$|\lambda_n - \mathbb{E}\left[f(W)\right]| \leq n^{-1+\gamma}.$$

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we have

$$\lambda_n := \mathbb{E}[D_n] = \mathbb{E}\Big[\frac{1}{n-1}\sum_{i=1}^{n-1} f(\deg_{n-1}^-(i))\Big] = \mathbb{E}\left[f(W_{n-1})\right].$$

Proof of Theorem 3: Stein's method

- Stein's equation

$$\mathbb{E}[\lambda g(Z+1) - Zg(Z)] = 0 \text{ for all bounded } g\colon \mathbb{N} \to \mathbb{R} \;\Leftrightarrow\; Z \sim Po(\lambda)$$

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- Barbour, Hall (1984):

$$d_{\mathrm{TV}}(D_n, Po(\lambda_n)) \leq \frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n-1} p_{i,n}^2 \leq \min\left\{1, \frac{1}{\lambda_n}\right\} \sum_{i=1}^{n-1} p_{i,n}^2.$$

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- here:

$$\sum_{i=1}^{n-1} p_{i,n}^2 = \frac{1}{(n-1)^2} \sum_{i=1}^{n-1} \mathbb{E}[f(\deg_{n-1}^{-}(i))]^2 \overset{(*)}{\leq} \frac{1}{(n-1)^2} \sum_{i=1}^{n-1} \left(\frac{n}{i}\right)^{2\gamma},$$

$(*)$: see Dereich, Mörters (2013).

Dynamic Random Networks

- joint projekt with Stephan Bussmann, Matthias Reitzner and from computer science Nils Aschenbruch

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- HUG, LAST, SCHULTE (2016):
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- Thank you!