Higher order approximation for sequences converging in the mod-Gaussian sense

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Symposium in Memory of Charles Stein

June 2019, Singapore

menu

- mod-Gauss convergence and some new results
- higher order approximation and some known results
- models with dependency and some new results

joint work with Lukas Knichel, Carolin Kleemann
and Marius Butzek, Bochum

mod-ϕ

- D a domain of $\mathbb{C}$ containing 0
- ϕ infinite divisible distribution with LAPLACE transform $\exp(\eta(z))$ on D η: LÉVY exponent

Definition (KOWALSKI, NIKEGHBALI, 2010)

A sequence of real r.v. (X_n) converges mod-ϕ on D with parameter $t_n \to \infty$ and limiting function ψ if, locally uniformly on D,

$$\exp(-t_n \eta(z))\,\mathbb{E}(e^{zX_n}) \to \psi(z)$$

instead of renormalizing the variables as in CLT, we renormalize the LAPLACE transform to get access to the next term

mod-ϕ implies a CLT

Theorem

If (X_n) *converges mod-*ϕ *on* D *with parameter* t_n*, then*

$$Y_n = \frac{X_n - t_n \eta'(0)}{\sqrt{t_n \eta''(0)}} \to N(0,1)$$

very often: classical way of proving CLTs can be adapted to prove mod-ϕ convergence

example 1

(Y_j) i.i.d., centered, $S_n = \sum_{j=1}^n Y_j$

$$\log \mathbb{E} e^{izS_n} = n \sum_{r \geq 2} \frac{\kappa^{(r)}(Y)}{r!} (iz)^r$$

$$X_n := \frac{S_n}{n^{1/\nu}}, \quad \nu \in \mathbb{N}$$

$$\log \mathbb{E} e^{izX_n} = -n^{\frac{\nu-2}{\nu}} \frac{\kappa^{(2)}(Y)}{2} z^2 + \sum_{j \geq \nu} \frac{\kappa^{(j)}(Y)}{j!} (iz)^j \frac{1}{n^{\frac{j}{\nu}-1}}$$

Hence if $n \to \infty$:

$$\frac{\mathbb{E} e^{izX_n}}{\exp\left(-n^{\frac{\nu-2}{\nu}} \sigma^2 \frac{z^2}{2}\right)} \to \exp\left(\frac{\kappa^{(\nu)}(Y)}{\nu!} (iz)^\nu \right) + \mathcal{O}\left(\frac{1}{n^{1/\nu}}\right)$$

example 1

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example 1

(X_n) converges mod-GAUSS mit parameter $t_n = \sigma^2 n^{\frac{\nu-2}{\nu}}$ and limiting function

$$\psi(z) = \exp\left(\frac{\kappa^{(\nu)}(Y)}{\nu!} (iz)^\nu \right)$$

example:

$\sigma^2 = 1$ and the law of Y is symmetric:

it could be that $\nu = 4$: $t_n = n^{1/2}$

and $\kappa^{(4)}(Y) = \mathbb{E} Y^4 - 3$

example 2

$$\log\big(|\det(Id - U_n)|\big)$$

where U_n is an unitary Haar-distributed random matrix

It converges mod-Gauss on $\{\mathrm{Re}(z) > -1\}$ with parameter $t_n = \frac{\log n}{2}$ and limiting function

$$\psi(z) = \frac{G(1+z/2)^2}{G(1+z)}$$

with $G(z+1) = G(z)\Gamma(z)$, $G(1) = 1$, the Barnes G-function

Kowalski, Nikeghbali, 2010; Keating, Snaith, 2000

example 3

Let M_n be a GUE matrix. Then

$$\log|\det M_n| - \mathbb{E}\log|\det M_n|$$

converges mod-Gauss on $\{|z| < 1\}$ with parameter $\frac{\log n}{2}$ and limiting function

$$\psi(z) = \frac{G(1+z/2)^2}{G(1+z)}$$

Döring, E., 2013; Dal Borgo, Hovhannisyan, Rouault, 2018

(we have studied uniform bounds on cumulants)

example 4: LAGUERRE ensemble

- Let A be a $n \times p(n)$ matrix with $p(n) \leq n$ choosen to be Gaussian over $\mathbb{C}$. Then $A^\dagger A$ is called LAGUERRE complex ensemble
- The joint pdf of $(\lambda_1, \ldots, \lambda_{p(n)})$ is

$$\frac{1}{Z_{n,p(n)}} \prod_{1 \leq j < k \leq p(n)} |\lambda_j - \lambda_k|^2 \prod_{k=1}^{p(n)} \lambda_k^{n-p(n)} e^{-\frac{\lambda_k}{2}}$$

- With SELBERG formula one obtains for the MELLIN transform

$$\mathbb{E}\left[\left(\det W^L_{n,p(n)}\right)^z\right] = 2^{p(n)z} \prod_{k=1+n-p(n)}^{n} \frac{\Gamma\left(k+z\right)}{\Gamma\left(k\right)}$$

example 4: Laguerre ensemble

- We obtain

$$\log \mathbb{E}\left[\exp(z \log(\det W^L_{n,p(n)}))\right] = p(n) z \log 2 + L(p(n), n; z)$$

- Applying Binet formula of $\log \Gamma(z)$, Barnes G-function and Abel-Plana summation formula we obtain

Theorem (E., Knichel, 2018)

mod-Gauss *of* $\log(\det W^L_{n,p(n)})$ *with parameters*

- $t_n = \log n$, *if* $p(n) = n$
- $t_n = \log\left(\frac{p(n)+1+c}{1+c}\right)$, *if* $n - p(n) = c$
- $t_n = \log\left(\frac{n}{n-p(n)}\right)$, *if* $n - p(n) = o(n)$

example 4: Laguerre ensemble

the limiting function is

$$\log \psi(z) = \log G(z+1) - (z - \frac{1}{2}) \log \Gamma(z+1) + \int_0^\infty \left(\frac{1}{2s} - \frac{1}{s^2} + \frac{1}{s(e^s - 1)} \right) \frac{e^{-sz} - 1}{e^s - 1} ds + \frac{3}{4} z^2 + \frac{z}{2}.$$

$p(n) = n$: Dal Borgo, Hovhannisyan, Rouault, 2018

Theorem (E., Knichel, 2018)

If $p(n) = p$, we obtain mod-ϕ convergence in $i\mathbb{R}$ with $t_n = p\, n$ and limiting function $\psi(z) = (1+z)^{-\frac{p^2}{2}}$, ϕ is an infinitely divisible distribution

example 4: Laguerre ensemble, companion theorems

- **extended CLT**: for $x = o(\sqrt{\log n})$

$$P\left(\log(\det W^L_{n,p(n)}) \geq x\sqrt{\log n}\right) = P\big(N(0,1) \geq x\big)\,(1 + o(1))$$

- **precise deviations**: for $x > 0$:

$$P\left(\log(\det W^L_{n,p(n)}) \geq x \log n\right) = \frac{e^{-\frac{x^2}{2}\log n}}{x\sqrt{2\pi \log n}}\psi(x)(1 + o(1))$$

- **rate of convergence**:

$$d_{\mathrm{Kol}}\left(\frac{\log(\det W^L_{n,p(n)})}{\sqrt{t_n}}, N(0,1)\right) \leq C\frac{1}{\sqrt{t_n}}$$

stochastic geometry

Random simplices:

- $p(n) + 1$ independent random points $X_1, \dots, X_{p(n)+1}$ in $\mathbb{R}^n$, multivariate Gaussian
- consider the $p(n)$-dimensional volume of the simplex with vertices $X_1, \dots, X_{p(n)+1}$
- dimension and number of points $p(n) + 1$ are allowed to grow simultaneously
- $\log \psi(z)$ differs by $-\frac{z^2}{2}$

see also GROTE, KABLUCHKO, THÄLE, 2018: cumulant bounds

higher order approximation

Consider a sequence of real r.v. (X_n) which converges mod-Gauss on $D \subset i\mathbb{R}$ with parameter $t_n \to \infty$ and limiting function ψ.

Theorem (Barhoumi-Andréani, 2017)

For nice ψ, one can construct a family of random variables $H_{t_n}(\psi)$ that converges mod-Gauss on D with parameter $t_n \to \infty$ and limiting function ψ.
The law of $H_{t_n}(\psi)$ has a Lebesgue-density given by:

$$\frac{1}{c_n}\psi\left(\frac{x}{t_n}\right)e^{-\frac{1}{2}\left(\frac{x}{\sqrt{t_n}}\right)^2}$$

higher order approximation

Question: can we find a bound for

$$d_{\mathrm{Kol}}\big(X_n, H_{t_n}(\psi)\big) \leq \quad ?$$

compare with the BERRY-ESSÉEN bound for the convergence in law of $\frac{X_n}{\sqrt{t_n}}$ to the Gaussian distribution!

BARHOUMI-ANDRÉANI, 2017:

consider on $D = i\mathbb{R}$ the limiting function $\psi(z) = e^{-C\frac{z^4}{4!}}$ as being the mod-GAUSSIAN limit of $X_n := \frac{1}{n^{1/4}} \sum_{j=1}^n Y_j$, i.i.d., with $t_n = n^{1/2}$:
Assume $\sigma^2 = 1$ **and that** $C := 3 - \mathbb{E}(Y^4) > 0$

higher order approximation: Stein's method

develop Stein's method for the target distribution with density

$$g_n(x) := \frac{1}{c_n} e^{-\frac{1}{2}\frac{x^2}{\sqrt{n}} - C\frac{x^4}{4!n^2}}$$

Barhoumi-Andréani, 2017

But: apply the **density approach** due to Stein to obtain the Stein equation and use the fact that the density is log-concave to apply results from Peköz, Röllin, Ross, 2016

$$f_h'(x) - \frac{x}{\sqrt{n}} f_h(x) - \frac{Cx^3}{6\,n^2} f_h(x) = h(x) - \mathbb{E}h$$

bound solutions for certain test-functions h ...

higher order approximation: results

Theorem (Barhoumi-Andréani, 2017)

Consider functions in

$$\mathcal{H}_1 := \{h \in C_1(\mathbb{R}) : \|h\|_\infty \leq 1, \|h'\|_\infty \leq 1\}$$

Then

$$d_{\mathcal{H}_1}\big(X_n, H_{\sqrt{n}}(\psi)\big) \leq \frac{C_1}{n^{1/4}}\|h'\|_\infty + \frac{C_2}{n^{1/2}}\|h\|_\infty$$

and the same order for the Kolmogorov*-distance*

Remark:

$$d_{\mathcal{H}_1}\left(\frac{X_n}{n^{1/4}}, \frac{H_{\sqrt{n}}(\psi)}{n^{1/4}}\right) \leq \frac{C_3}{\sqrt{n}} + \mathcal{O}\big(\frac{1}{n}\big)$$

higher order approximation: results

Theorem (Barhoumi-Andréani, 2017)

Consider functions in

$$\mathcal{H}_2 := \{h \in C_1(\mathbb{R}) : \|h\|_\infty \leq 1, \|h'\|_\infty \leq 1, \|\mathbf{h}''\|_\infty \leq \mathbf{1}\}$$

Then

$$d_{\mathcal{H}_2}\big(X_n, H_{\sqrt{n}}(\psi)\big) \leq \frac{C_1}{\sqrt{n}} + \frac{C_2}{n^{\frac{3}{4}}}$$

Now:

$$d_{\mathcal{H}_2}\left(\frac{X_n}{n^{1/4}}, \frac{H_{\sqrt{n}}(\psi)}{n^{1/4}}\right) \leq \frac{C_3}{n} + \mathcal{O}\big(\frac{1}{n^2}\big)$$

$$d_{Kol}\left(\frac{X_n}{n^{1/4}}, \frac{H_{\sqrt{n}}(\psi)}{n^{1/4}}\right) \leq \mathcal{O}\big(\frac{1}{n^{2/3}}\big)$$

higher order approximation: results

improvement of the classical Berry-Esseen bound understood as an additive correction:

$$\left| \mathbb{E}h\left(\frac{X_n}{n^{1/4}}\right) - Eh\left(\frac{H_{\sqrt{n}}}{n^{1/4}}\right)\right| = \left|\mathbb{E}h\left(\frac{X_n}{n^{1/4}}\right) - \mathbb{E}(h(N)) + \mathrm{Corr}(n,h)\right|$$

with

$$\mathrm{Corr}(n,h) = \mathbb{E}\left(h(N) - h\left(\frac{H_{\sqrt{n}}}{n^{1/4}}\right)\right) = \mathbb{E}\left(h'(\frac{H_{\sqrt{n}}}{n^{1/4}} + U\Delta_n)\Delta_n\right)$$

where U uniform, N and $H_{\sqrt{n}}$ are independent and $\Delta_n := N - \frac{H_{\sqrt{n}}}{n^{1/4}}$

higher order approximation

Challanges:

- one has to bound the **third** derivative of STEIN-solutions !
- one has to care on the **perturbative operator**

$$\frac{C}{6n^2}\,\mathbb{E}\big(X_n^3 f(X_n)\big)$$

(B.-A.: zero-bias techniques...)

goal: non-identically distributed and dependent variables

Questions: try to proof similar results

1. for independent but **non-identically** distributed summands
2. for **exchangeable pairs** (application: Curie-Weiss model with $\beta \neq \beta_c$)
3. for summands with a **dependency graph** structure
4. other (t_n, ψ) mod-Gauss situations...

1. **non-identically** distributed

(Y_j) independent, centered, use exchangeable pair-approach

$$X_n = \frac{1}{n^{\frac{1}{4}}} \sum_{j=1}^n Y_j, \qquad \mathbb{E}(X_n' | X_n) = \big(1 - \frac{1}{n}\big) X_n$$

$$\big| \mathbb{E}\big(f'(X_n) - \frac{X_n}{\sqrt{n}} f(X_n)\big) \big| \leq \frac{C_1}{n^{3/2}} \sum_{j=1}^n \mathbb{E}(Y_j^4) + \frac{C_2}{n} \left(\sum_{j=1}^n \mathrm{Var}(Y_j^2) \right)^{1/2}$$

1. **non-identically** distributed

with $\tilde{f}(x) := x^2 f(x)$, $\lambda = \frac{1}{n}$, $\mathbb{E}(X_n'|X_n) = (1-\lambda)X_n$ and

$$\widehat{K}(t) := (X_n - X_n')\big(1_{\{-(X_n - X_n') \le t \le 0\}} - 1_{\{0 \le t \le -(X_n - X_n')\}}\big)$$

we obtain

$$\frac{C}{6n^2}\mathbb{E}\big(X_n^3 f(X_n)\big) = \frac{C}{6n^2}\mathbb{E}\big(X_n \tilde{f}(X_n)\big) = \frac{C}{6n^2}\frac{1}{\lambda}\mathbb{E}\int_{\mathbb{R}} \tilde{f}'(X_n + t)\widehat{K}(t)\,dt$$

second equality well known...

1. **non-identically** distributed

notice that $\tilde{f}'(X_n + t) = (X_n + t)^2 f'(X_n + t) + 2(X_n + t) f(X_n + t)$

after some calculations we obtain:

$$\left| \frac{C}{6n} \mathbb{E} \int_{\mathbb{R}} \tilde{f}'(X_n + t) \widehat{K}(t)\, dt \right| \leq \frac{C_1}{n^3} \sum_{j=1}^{n} \mathbb{E} Y_j^4 + \frac{C_2}{n} + \frac{C_3}{n^{5/2}} \sum_{j=1}^{n} \mathbb{E}|Y_j|^3 + \frac{C_4}{n^{3/2}} \sum_{j=1}^{n} \mathbb{E}|Y_j|$$

hence we have obtained the aimed general bound for $d_{\mathcal{H}_2}$ (and $d_{\mathcal{H}_1}$)

2. Curie-Weiss model

n magnetic particles $Y_1, \ldots, Y_n$: each with spin $+1, -1$
density:

$$\frac{1}{Z_n} \exp\left(\frac{\beta}{2n} \sum_{i,j=1}^{n} y_i \, y_j \right) d\varrho^{\otimes n}(y)$$

with $\varrho = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$

the particles try to align themselves together with the same spin, β inverse temperature

Theorem (Ellis, Newman, 1978)

for $0 < \beta < 1$ the magnetization $\frac{1}{n}\sum Y_j$ fulfils:

$$W_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j \Rightarrow N\big(0, \frac{1}{1-\beta}\big)$$

$\beta = 1$*: critical temperature (phase transition)*

2. Stein-pair for Curie-Weiss

$$W_n' := W_n + \frac{Y_I' - Y_I}{\sqrt{n}}$$

$$\mathbb{E}(W_n'|W_n) = \big(1 - \frac{1-\beta}{n}\big) W_n + R = \big(1 - \lambda\big) W_n + R$$

known (Barbour 1980, Chatterjee, Shao 2010; E., Löwe 2010):
for $0 < \beta < 1$:

$$\sup_{z \in \mathbb{R}} |\mathbb{P}_{CW}(W_n \leq z) - \Phi_\beta(z)| \leq \frac{C}{\sqrt{n}}$$

2. Curie-Weiss model

Theorem (Butzek, E., 2019)

Consider the Curie-Weiss *model and let* $X_n := n^{1/4} W_n$.
Take $t_n = \sqrt{n}$, $\psi(z) = e^{-Cz^4}$ *and any* $0 < \beta < 1$:

$$d_{\mathcal{H}_2}\left(X_n, H_{\sqrt{n}}(\psi)\right) \leq \frac{C_1}{\sqrt{n}} + \frac{C_2}{n^{\frac{3}{4}}}$$

and

$$d_{\mathcal{H}_2}\left(W_n, \frac{H_{\sqrt{n}}(\psi)}{n^{1/4}}\right) \leq \frac{C}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \qquad d_{Kol}\left(W_n, \frac{H_{\sqrt{n}}(\psi)}{n^{1/4}}\right) \leq \frac{C}{n^{2/3}}$$

2. sketch of the proof

we apply a recent bound due to Shao, Zhang, 2018:

$$\mathbb{E}\left|\frac{1}{2\lambda}\mathbb{E}\big((W_n - W_n')\,|W_n - W_n'|\big|W_n\big)\right| \leq C\frac{1}{\sqrt{n}}$$

to obtain

$$\big|\mathbb{E}\big(f'(X_n) - \frac{X_n}{\sqrt{n}}f(X_n)\big)\big| = \mathcal{O}\big(\frac{1}{\sqrt{n}}\big)$$

again:

$$\begin{aligned}
\frac{C}{6n^2}\mathbb{E}\big(X_n^3 f(X_n)\big) &= \frac{C}{6n^2}\mathbb{E}\big(X_n\tilde{f}(X_n)\big) \\
&= \frac{C}{6n^2}\frac{1}{2\lambda}\mathbb{E}\int_{\mathbb{R}} \tilde{f}'(X_n+t)\widehat{K}(t)\,dt + \frac{C}{6n^2}\frac{1}{\lambda}\mathbb{E}\big(\tilde{f}(X_n)R(X_n)\big) \\
&= \mathcal{O}\big(\frac{1}{\sqrt{n}}\big)
\end{aligned}$$

3. dependency graphs

a graph L with vertex set A is a dependency graph for $(Y_\alpha)_{\alpha \in A}$ if

- if A_1 and A_2 are disconnected subsets in L, then $(Y_\alpha)_{\alpha \in A_1}$ and $(Y_\alpha)_{\alpha \in A_2}$ are independent
- roughly: there are edges between pairs of *dependent* random variables
- example: $G(n, p)$-model, counting Δs; edge between two, if they share an edge

3. dependency graph

setting:

- $(Y_{n,i})_{1 \le i \le N_n}$ family of bounded random variables, $|Y_{n,i}| < A$.
- there is a dependency graph L_n with maximal degree $D_n - 1$
- consider $S_n = \sum_{i=1}^{N_n} Y_{n,i}$ and $\sigma_n^2 = \mathbb{V}(S_n)$

JANSON, 1988: $\kappa_r(S_n) \le C_r N_n D_n^{r-1} A^r$, C_r a constant

assume that $\frac{\sigma_n^2}{D_n N_n} \to \sigma^2 > 0$: error term in the CLT is $\mathcal{O}\left(\sqrt{\frac{D_n}{N_n}}\right)$

3. dependency graph

Theorem (E., Kleemann, 2019)

Assume that $\frac{\sigma_n^2}{N_n D_n} \to \sigma^2 > 0$ *and* $\frac{\kappa_4(S_n)}{N_n D_n^3} \to K > 0$*. Moreover assume that* $\mathbb{E} Y_{n,i}^3 = 0$ *for all* n, i*. Then*

$$X_n := \frac{S_n - \mathbb{E}(S_n)}{D_n^{3/4} N_n^{1/4}}$$

converges mod-Gauß with $t_n = \left(\frac{N_n}{D_n}\right)^{1/2}$ *and limiting function* $\psi(z) = \exp(-\frac{K}{4} z^4)$*. Moreover*

$$d_{\mathcal{H}_1}(X_n, H_{t_n}(\psi)) \leq \mathcal{O}\left(\left(\frac{D_n}{N_n}\right)^{1/4}\right).$$

needs good bounds on cumulants!

thank you