# Central limit theorems on Wiener chaoses

## Symposium in memory of Charles Stein

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Stein's method using kernels: Cacoullos-Papathanasiou-Utev (1994)

- $\mu$ is a probability measure
- $\tau$ is a Borel map in $L^1(\mu)$
- $\forall \phi \in \mathscr{C}_b^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \phi'(x)\tau(x)d\mu_x = \int_{\mathbb{R}} x\phi(x)d\mu_x$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$d_{TV}(\mu,\gamma) \le 2\int_{\mathbb{R}} \left|1-\tau(x)\right| d\mu_x$$

Malliavin-Stein's method: Peccati-Nourdin (2009)

- $(\Omega, \mathscr{F}, \mathbb{P})$ probability space
- $\Gamma$ the square field operator
- $\mathscr{L}$ the Ornstein-Uhlenbeck operator
- $\forall \phi \in \mathscr{C}_b^1(\mathbb{R}), \forall X \in \text{dom}(\Gamma)$:

$$\mathbb{E}(\phi'(X)\Gamma[X, -\mathscr{L}^{-1}X]) = \mathbb{E}(\phi(X)X)$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$d_{TV}(X,N) \le 2\ \mathbb{E}(\left|1-\Gamma[X,-\mathscr{L}^{-1}X]\right|)$$

# Central convergence on second Wiener chaos

The setup:

- $F = \sum_{i,j=1}^{m} a_{i,j} X_i X_j$,
- $(X_1, \cdots, X_m)$ are i.i.d. $\mathcal{N}(0,1)$,
- $A = (a_{i,j})_{1 \le i,j \le m}$ is a real symmetric matrix,
- $\mathrm{Diag}(A) = 0$,
- $\mathbb{E}(F^2) = 1$.

Diagonalizing the matrix $A$ leads to:

$$F = \sum_{i=1}^{m} \alpha_i Y_i^2$$

- $(Y_1, \cdots, Y_m) = (X_1, \cdots, X_m) P \sim \mathcal{N}(0, I_m) \quad (\text{since } P \in O_m(\mathbb{R}))$
- $(\alpha_1, \cdots, \alpha_m) = \mathrm{Spec}(A)$.

In order to use Malliavin Lemma, we need to find $p \geq 1$ such that $\frac{1}{\Gamma[F,F]} \in L^p(\mathbb{P})$.

$$\begin{aligned}\Gamma[F,F] &= \sum_{i,j=1}^{m} \alpha_i \alpha_j \Gamma[Y_i^2, Y_j^2] \\ &\overset{\text{C.R.}}{=} 4 \sum_{i,j=1}^{m} \alpha_i \alpha_j Y_i Y_j \Gamma[Y_i Y_j]\end{aligned}$$

$$\begin{aligned}\Gamma[Y_i, Y_j] &= \sum_{k,l=1}^{m} P_{k,i} P_{l,j} \Gamma[X_k, X_l] \\ &= \sum_{k=1}^{m} P_{k,i} P_{k,j} = [{}^t PP]_{i,j} = \underbrace{\delta_{i,j}}_{\text{Kronecker symbol}}\end{aligned}$$

$$\Gamma[F,F] = 4 \sum_{i=1}^{m} \alpha_i^2 Y_i^2$$

For every $\lambda > 0$ we have

$$\begin{aligned}\mathbb{E}(\exp(-\lambda\Gamma[F,F])) &= \prod_{i=1}^{m}\mathbb{E}\left(\exp\left(-\lambda\alpha_i^2 Y_i^2\right)\right)\\ &= \prod_{i=1}^{m}\frac{1}{\sqrt{1+8\lambda\alpha_i^2}}\end{aligned}$$

On the other hand,

$$\begin{aligned}\prod_{i=1}^{m}(1+8\lambda\alpha_i^2) &= 1+\sum_{j=1}^{m}8^j\lambda^j S_j\\ S_j &= \sum_{1\le k_1<k_2<\cdots<k_j\le m}\alpha_{k_1}^2\cdots\alpha_{k_p}^2\end{aligned}$$

Recall that Nualart-Peccati criterion asserts that

$$\begin{aligned} F \overset{\text{Law}}{\approx} \mathcal{N}(0,1) \quad & \Leftrightarrow \quad \mathbb{E}\left(F^4\right) \approx 3 \\ & \Leftrightarrow \quad \sum_{k=1}^{m} \alpha_k^4 \ll \approx 0 \\ & \Leftrightarrow \quad \underbrace{\max_{1 \le k \le m} |\alpha_k|}_{\delta_F} \approx 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} S_j &= \sum_{1 \le k_1 < k_2 < \cdots < k_j \le m} \alpha_{k_1}^2 \cdots \alpha_{k_p}^2 \\ &= \frac{1}{p!} \sum_{k_1 \neq k_2 \cdots \neq k_p} \alpha_{k_1}^2 \cdots \alpha_{k_p}^2 \\ &\geq \frac{S_1(S_1 - \delta_F) \cdots (S_1 - (j-1)\delta_F)}{p!} \end{aligned}$$

Let us summarize:

- $F \overset{\text{Law}}{\approx} \mathcal{N}(0,1)$ if and only if $\delta_F$ is small
- for all $j \geq 2$, $S_j \geq \frac{1}{p!} S_1(S_1 - \delta_F) \cdots (S_1 - (j-1)\delta_F)$,
- $S_1 = \frac{1}{2}\mathbb{E}(F^2) = \frac{1}{2}$,
- $$\mathbb{E}(\exp(-\lambda\Gamma[F,F])) = \frac{1}{\sqrt{1 + \sum_{j=1}^m 8^j \lambda^j S_j}}.$$

Combining these facts leads to

$$\forall q \geq 1, \exists \epsilon_q, C_q > 0, \text{ such that:}$$

$$d_{TV}(F, \mathcal{N}(0,1)) < \epsilon_q \Rightarrow \mathbb{E}(\exp(-\lambda\Gamma[F,F])) < \frac{C_q}{\lambda^q}.$$

Assume that we have

$$\mathbb{E}(\exp(-\lambda\Gamma[F,F])) < \frac{C_q}{\lambda^q}.$$

Take $\alpha > 0$, one gets

$$\begin{aligned}\mathbb{P}(\Gamma[F,F] < \alpha) &= \mathbb{P}\left(\exp(-\lambda\Gamma[F,F]) > e^{-\lambda\alpha}\right)\\ &\leq e^{\lambda\alpha}\mathbb{E}(\exp(-\lambda\Gamma[F,F]))\\ &\leq \frac{C_q}{\lambda^q}e^{\lambda\alpha}\end{aligned}$$

Chosing $\lambda = \frac{1}{\alpha}$ leads to

$$\mathbb{P}(\Gamma[F,F] < \alpha) < C_q\alpha^q \Rightarrow \frac{1}{\Gamma[F,F]} \in L^{q-}(\mathbb{P}).$$

## G.P. (2019)

Let $F \in \text{Ker}(\mathscr{L} + 2\mathbf{I})$ which satisfies $\mathbb{E}(F^2) = 1$. One has, for every $p \geq 1$,

$$\kappa_4(F) < \frac{24}{4^p(2p+1)!} \Rightarrow \frac{1}{\Gamma[F,F]} \in L^p(\Omega, \mathscr{F}, \mathbb{P}).$$

- What is the correct speed of "regularization"?
- Does this phenomenon extend to chaoses of higher order?

# Proof of a weaker Carbery-Wright estimate

- $P \in \mathbb{R}_d[x_1, \cdots, x_m]$,
- $\mathbf{X} = (X_1, \cdots, X_m)$ **i.i.d.** $\sim \mathcal{N}(0,1)$,
- $\mathbb{E}(P(\mathbf{X})^2) = 1$.

We proceed by induction on the degree $d$.

$$\begin{aligned}
\mathbb{P}\left(\left|P(\mathbf{X})\right| < \epsilon\right) &= \mathbb{E}\left(\mathbf{1}_{[-\epsilon,\epsilon]}\left(P(\mathbf{X})\right)\right) \\
&= \mathbb{E}\left(\mathbf{1}_{[-\epsilon,\epsilon]}\left(P(\mathbf{X})\right) \frac{\Gamma[P(\mathbf{X}), P(\mathbf{X})]}{\Gamma[P(\mathbf{X}), P(\mathbf{X})]}\right) \\
&\leq \underbrace{\frac{1}{\delta}\mathbb{E}\left(\mathbf{1}_{[-\epsilon,\epsilon]}\left(P(\mathbf{X})\right)\Gamma[P(\mathbf{X}), P(\mathbf{X})]\right)}_{\text{A}} \\
&+ \underbrace{\mathbb{P}\left(\Gamma[P(\mathbf{X}), P(\mathbf{X})] \leq \delta\right)}_{\text{B}}
\end{aligned}$$

We introduce $\chi_\epsilon(x) = \int_0^x \mathbf{1}_{[-\epsilon,\epsilon]}(t)dt$:

$$\begin{aligned} A &= \frac{1}{\delta}\mathbb{E}\left(\chi'_\epsilon\left(P(\mathbf{X})\right)\Gamma[P(\mathbf{X}), P(\mathbf{X})]\right) \quad \text{C.R.} \\ &= \frac{1}{\delta}\mathbb{E}(\Gamma[\chi_\epsilon(P(\mathbf{X})), P(\mathbf{X})]) \\ &= -\mathbb{E}(\chi_\epsilon(P(\mathbf{X}))\mathscr{L}P(\mathbf{X})) \\ &\leq \frac{\epsilon}{\delta}\ \mathbb{E}(\left|\mathscr{L}P(\mathbf{X})\right|) \\ &\leq \frac{\epsilon}{\delta}\ d\ \sqrt{\mathbb{E}(P(\mathbf{X})^2)} \\ &= \frac{\epsilon}{\delta}\ d \end{aligned}$$

$$
\begin{aligned}
{}^{\sharp}P(\mathbf{X}) &= \sum_{i=1}^{m} Y_i \frac{\partial P(\mathbf{X})}{\partial X_i} \\
&\overset{\text{Law}}{=} N\sqrt{\Gamma[P(\mathbf{X}),P(\mathbf{X})]}
\end{aligned}
$$

with $N \sim \mathcal{N}(0,1)$ independent of $\mathbf{X}$. Using induction assumption, **conditionnally to Y** gives:

$$
\begin{aligned}
\mathbb{P}\left(\left|{}^{\sharp}P(\mathbf{X})\right| < \delta\right) &= \mathbb{P}\left(\frac{\left|{}^{\sharp}P(\mathbf{X})\right|}{\sqrt{\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2)}} < \frac{\delta}{\sqrt{\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2)}}\right) \\
&\leq C_{d-1}\ \delta^{\alpha_{d-1}}\ \underbrace{\mathbb{E}_{\mathbf{Y}}\left(\frac{1}{\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2)^{\frac{\alpha_{d-1}}{2}}}\right)}_{C}
\end{aligned}
$$

- By Poincaré inequality, $\mathbb{E}_{\mathbf{Y}}\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2) = \mathbb{E}(\Gamma[P(\mathbf{X}), P(\mathbf{X})]) \geq 1$,
-

$$\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2) = \underbrace{\sum_{i,j=1}^{m} Y_i Y_j \mathbb{E}\left(\frac{\partial P(\mathbf{X})}{\partial X_i}\frac{\partial P(\mathbf{X})}{\partial X_j}\right)}_{\text{positive quadratic form of } \mathbf{Y}}$$

$$\overset{\text{Law}}{=} \sum_{i=1}^{m} \underbrace{\alpha_i}_{\geq 0} Y_i^2 \quad \Rightarrow \quad \sum_{i=1}^{m} \alpha_i \geq \frac{1}{2}.$$

- Computing Laplace transform one gets

$$\mathbb{P}\left(\sum_{i=1}^{m} \alpha_i Y_i^2 < \epsilon\right) < \sqrt{\epsilon}$$

$$\Rightarrow \mathbb{E}_{\mathbf{Y}}\left(\frac{1}{\mathbb{E}_{\mathbf{X}}({}^{\sharp}P(\mathbf{X})^2)^{\frac{\alpha_{d-1}}{2}}}\right) < 1$$

$$\Rightarrow \quad \mathbb{P}\left(\left|{}^{\sharp}P(\mathbf{X})\right| < \delta\right) < C_{d-1}\delta^{\alpha_{d-1}}.$$

Let summarize

- $\mathbb{P}\left(\left|P(\mathbf{X})\right| < \epsilon\right) < d\ \frac{\epsilon}{\delta} + \mathbb{P}\left(\Gamma[P(\mathbf{X}), P(\mathbf{X})] < \delta\right)$,
- $\mathbb{P}\left(\left|{}^{\sharp}P(\mathbf{X})\right| < \delta\right) < C_{d-1}\delta^{\alpha_{d-1}}$

We notice that ${}^{\sharp}P(\mathbf{X}) \overset{\text{Law}}{=} N\sqrt{\Gamma[P(\mathbf{X}), P(\mathbf{X})]}$ with $N \models \Gamma[P(\mathbf{X}), P(\mathbf{X})]$.

One can deduce that

$$\mathbb{P}\left(\Gamma[P(\mathbf{X}), P(\mathbf{X})] < \delta\right) < \frac{C_{d-1}}{\mathbb{P}\left(|N| < 1\right)}\ \delta^{\frac{1}{2}\alpha_{d-1}}$$

Optimization gives for $P$ of degree $d$ such that $\mathbb{E}\left(P(\mathbf{X})^2\right) = 1$ that:

$$\mathbb{P}\left(\left|P(\mathbf{X})\right| < \epsilon\right) \leq \frac{1}{\mathbb{P}\left(|N| < 1\right)^2} d\ \ \epsilon^{\frac{1}{2^d}}.$$